



DEPARTMENT OF MATHEMATICS & SC

**ENGINEERING MATHEMATICS-II
FOR DIPLOMA 2ND SEMESTER**

LEARNING MATERIAL

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Unit-1 - Vector

Note:-

Q.1. What is Vector?

> A physical quantity with specified magnitude as well as definite direction.

2. How Vector Represented?

> A vector generally represented by a directed line segment, say $\vec{AB}$.

> 'A' is called initial point and B is called the terminal point.

3. Zero Vector

> A vector of zero magnitude is a zero vector.
i.e. which has the same initial & terminal point,

> It is denoted as " $\vec{0}$ ".

> The direction of zero vector is indeterminate.
i.e. infinitely many direction

> Zero vector is also called null vector.

4. Unit Vector

> Vector of unit magnitude

> $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along x, y, & z-axis.

> Unit vector along any vector $\vec{a}$ is denoted by $\hat{a}$ and $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

5. Equal Vector

Two vectors are said to be equal if they have the same magnitude, direction & represent the same physical quantity.

6. Position vector of a point

> Let O be a fixed origin, then the position vector of a point P is the vector $\vec{OP}$.

> If $\vec{a}$ & $\vec{b}$ are position vectors of two points A and B , then $\vec{AB} = \vec{b} - \vec{a}$.

7. Addition of vectors

If two vectors $\vec{a}$ and $\vec{b}$ are represented by $\vec{OA}$ & $\vec{OB}$, then their sum $\vec{a} + \vec{b}$ is a vector represented by $\vec{OC}$, where OC is the diagonal of the parallelogram $OACB$.

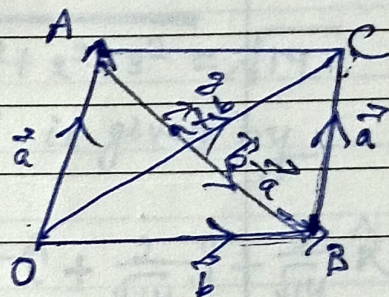
Properties

> $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative)

> $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (associative)

> $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$

> $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$



8. Multiplication of a vector by a scalar

If $\vec{a}$ is a vector & m is a scalar; then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times than $\vec{a}$, this multiplication is called scalar multiplication. If $\vec{a}$ & $\vec{b}$ are vectors

& m, n are scalar, then,

$$m(\vec{a}) = (\vec{a})m = m\vec{a}$$

$$m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$$

$$(m+n)\vec{a} = m\vec{a} + n\vec{a}$$

$$m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$$

9. Representation of vector in Component form

Let O be the origin and let $P(x, y, z)$ be any point in space. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along x, y , & z -axis respectively. Let the position vector of P be $\vec{r}$,

then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

x, y, z , are called scalar components of $\vec{r}$

$x\hat{i}, y\hat{j}, z\hat{k}$ are called its vector components

10. Magnitude of vector

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Solved Example

Example - 1

Find a unit vector in the direction of the vector $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Solution:- $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k} \Rightarrow |\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

Unit Vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$

Example - 2

Find a unit vector in the direction of $\vec{AB}$, where $A(1, 2, 3)$ and $B(4, 5, 6)$ are given points.

Solution:-

we have position vector of $A = \hat{i} + 2\hat{j} + 3\hat{k}$

position vector of $B = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$$\begin{aligned} \therefore \vec{AB} &= (\text{p.v. of } B) - (\text{p.v. of } A) \\ &= (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) \\ &= 3\hat{i} + 3\hat{j} + 3\hat{k} \end{aligned}$$

$$|\vec{AB}| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{27}$$

$$\begin{aligned} \therefore \text{Unit vector in the direction of } \vec{AB} &= \frac{\vec{AB}}{|\vec{AB}|} \\ &= \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{\sqrt{27}} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}} = \frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}} \end{aligned}$$

Example 3

Q1. If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$, find a unit vector in the direction of $\vec{a} + \vec{b}$.

Solution: Let $\vec{c} = \vec{a} + \vec{b}$, then

$$\begin{aligned}\vec{c} &= (\hat{i} + 2\hat{j} - 3\hat{k}) + (2\hat{i} + 3\hat{j} + \hat{k}) \\ &= (3\hat{i} + 5\hat{j} - 2\hat{k})\end{aligned}$$

$$|\vec{c}| = \sqrt{3^2 + 5^2 + (-2)^2} = \sqrt{38}$$

Hence, the required vector is given by

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{3\hat{i} + 5\hat{j} - 2\hat{k}}{\sqrt{38}} = \frac{3}{\sqrt{38}}\hat{i} + \frac{5}{\sqrt{38}}\hat{j} - \frac{2}{\sqrt{38}}\hat{k}$$

Assignment - 1

Q1. Write down the magnitude and a unit vector in the direction of the following vectors.

(i) $\vec{a} = \hat{i} + 2\hat{j} + 5\hat{k}$ (ii) $\vec{b} = 5\hat{i} - 4\hat{j} - 3\hat{k}$

Q2. Show that the points A, B, C with position vector

$\vec{a} = (3\hat{i} - 9\hat{j} - 4\hat{k})$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$ respectively form the vertices of a right-angled triangle.

Q3. Find the unit vector in the direction of the vector $\vec{a} + 2\vec{b} - \vec{c}$ if $\vec{a} = \hat{i} - 3\hat{j} - 5\hat{k}$, $\vec{b} = 3\hat{i} + 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + \hat{k}$.

11. Scalar Product of Vectors (Dot Product)

Let $\vec{a}$ and $\vec{b}$ be two vectors and let θ be the angle between them, then the scalar product, or dot product, of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$.

N.B.: the scalar product of two vectors is a scalar.

12. Angle between two vectors in terms of scalar product

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \Rightarrow \theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$$

13. N.B.: (i) If $\vec{a}$ and $\vec{b}$ are like vectors $\theta = 0$

$$\vec{a} \cdot \vec{b} = ab \cos 0 = ab \quad (\cos 0 = 1)$$

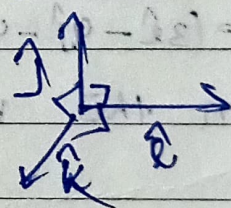
$$|\vec{a} \cdot \vec{a}| = a^2$$

$$(ii) \text{ If } |\vec{a} \perp \vec{b}|, \theta = 90^\circ \Rightarrow \vec{a} \cdot \vec{b} = 0$$

$$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b} \cos 90^\circ = 0$$

$$(iii) \begin{cases} \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = \hat{k} \cdot \hat{i} = 0 \\ \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \end{cases}$$

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$



14. Working Rule

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\vec{a} \cdot \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= a_1 \hat{i} \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) + a_2 \hat{j} \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) + a_3 \hat{k} \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= a_1 b_1 \hat{i} \cdot \hat{i} + a_2 b_2 \hat{j} \cdot \hat{j} + a_3 b_3 \hat{k} \cdot \hat{k}$$

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

Examples

Q1. Find the angle between the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$.

Solution :- $|\vec{a}| = \sqrt{3^2 + (-2)^2 + (1)^2} = \sqrt{14}$

$$|\vec{b}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{14}$$

Now $\vec{a} \cdot \vec{b} = 3 \cdot 1 + (-2)(-2) + (1)(3)$

$$= 3 + 4 + 3 = 10$$

$$\theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right) = \cos^{-1} \left(\frac{10}{\sqrt{14} \sqrt{14}} \right) = \cos^{-1} \left(\frac{10}{14} \right)$$

$$\Rightarrow \boxed{\theta = \cos^{-1} \frac{5}{7}}$$

Q2. Find the value of λ for which the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} + \lambda\hat{j} - 3\hat{k}$ are $\perp$ to each other.

Solution:- $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$

$$\Rightarrow (3\hat{i} + \hat{j} - 2\hat{k}) \cdot (C\hat{i} + \lambda\hat{j} - 3\hat{k}) = 0$$

$$\Rightarrow 3 \times 1 + 1 \times \lambda + (-2)(-3) = 0$$

$$\Rightarrow 3 + \lambda + 6 = 0$$

$$\Rightarrow \lambda + 9 = 0$$

$$\Rightarrow \boxed{\lambda = -9}$$

15. Scalar project and vector projection

① scalar projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ (scalar quantity)

④ vector projection of $\vec{a}$ on $\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b}$.

$$M.B = [(S.P) \text{ of } \vec{a} \text{ on } \vec{b}] \times \hat{b} = V.P \text{ of } \vec{a} \text{ on } \vec{b}$$

Example 1:-

Find the projection of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

Solution:- scalar projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$\vec{a} \cdot \vec{b} = (2\hat{i} + 3\hat{j} + 2\hat{k}) \cdot (\hat{i} + 2\hat{j} + \hat{k})$$

$$= 2 \times 1 + 3 \times 2 + 2 \times 1$$

$$= 2 + 6 + 2$$

$$= 10$$

$$|\vec{b}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\text{s.p} = \frac{10}{\sqrt{6}}$$

$$\text{vector projection of } \vec{a} \text{ on } \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \right) \hat{b}$$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b}$$

$$= \frac{10}{(\sqrt{6})^2} (\hat{i} + 2\hat{j} + \hat{k})$$

$$= \frac{10}{6} (\hat{i} + 2\hat{j} + \hat{k})$$

$$= \frac{5}{3} (\hat{i} + 2\hat{j} + \hat{k})$$

$$\text{vector projection of } \vec{a} \text{ on } \vec{b} = \frac{5}{3}\hat{i} + \frac{10}{3}\hat{j} + \frac{5}{3}\hat{k}$$

Assignment - 2

- Q1. Find the value of λ for which $\vec{a}$ & $\vec{b}$ are perpendicular, where $\vec{a} = 3\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = -\lambda\hat{i} + 3\hat{j} + 3\hat{k}$.
- Q2. Find the angle between the vectors $\vec{a}$ and $\vec{b}$ where, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.
- Q3. Let $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$, find the projection of (i) $\vec{a}$ on $\vec{b}$ and (ii) $\vec{b}$ on $\vec{a}$.

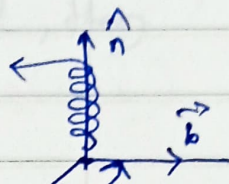
16. Vector (cross) product

Let $\vec{a}$ and $\vec{b}$ be two non-zero, non-parallel vectors, and let θ be the angle between them such that $0 < \theta < \pi$.

Then, the vector product is defined as

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$$

$\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}, \hat{n}$ form a right-handed system.



17. Sine Angle between two vectors

Let θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta \quad [\because |\hat{n}| = 1 \text{ as a unit vector}]$$

$$\Rightarrow \sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$$

18. Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

19. Properties

$$(1) \vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}) \quad (\text{not commutative})$$

$$(2) m(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$$

$$(3) \vec{a} \parallel \vec{b} \Rightarrow \vec{a} \times \vec{b} = 0$$

$$\text{N.B.} \quad \begin{aligned} \hat{i} \times \hat{j} &= \hat{j} \times \hat{k} = \hat{k} \times \hat{i} = \hat{1} \\ \hat{i} \times \hat{i} &= \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \end{aligned}$$

20. Application of cross product

(i) Area of parallelogram with two adjacent sides $\vec{a}$ & $\vec{b}$ is given by $|\vec{a} \times \vec{b}|$ sq units

(ii) Area of a parallelogram with diagonal $\vec{c}$ & $\vec{d}$ is $\frac{1}{2} |\vec{c} \times \vec{d}|$ sq units

(iii) Area of ΔABC ; where $\vec{AB} = \vec{c}$, $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ is $\frac{1}{2} |\vec{a} \times \vec{b}|$ or $\frac{1}{2} |\vec{b} \times \vec{c}|$ or $\frac{1}{2} |\vec{c} \times \vec{a}|$ sq units.

21. Working Rule of vector product (in terms of components)

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Solved Examples

Example-1 If $\vec{a} = 3\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 6\hat{i} + 5\hat{j} - 2\hat{k}$.
Find $\vec{a} \times \vec{b}$ and $|\vec{a} \times \vec{b}|$.

Solution 1-

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -4 \\ 6 & 5 & -2 \end{vmatrix}$$

$$= (-2 + 20)\hat{i} - (-6 + 24)\hat{j} + (15 - 6)\hat{k}$$

$$= 18\hat{i} - 18\hat{j} + 9\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{18^2 + (-18)^2 + 9^2} = \sqrt{729} = 27$$

Example-2 $\therefore$ find a Unit vector $\perp^r$ to each one of the vectors $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 2\hat{j} - \hat{k}$.

Solution 1-

We know that $\vec{a} \times \vec{b}$ is a vector $\perp^r$ to each one of $\vec{a}$ and $\vec{b}$.

so Unit vector $\perp^r$ to both $\vec{a}$ & $\vec{b}$ is $\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$\text{Now } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ 2 & 2 & -1 \end{vmatrix} = (1-6)\hat{i} - (-4-6)\hat{j} + (8+2)\hat{k} \\ = -5\hat{i} + 10\hat{j} + 10\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-5)^2 + (10)^2 + (10)^2} = \sqrt{225} = 15$$

Hence, the required unit vector $= \frac{-5\hat{i} + 10\hat{j} + 10\hat{k}}{15}$

$$= \frac{5}{15}(-\hat{i} + 2\hat{j} + 2\hat{k})$$

$$= \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$$

Example-3

Find the sine angle between the vectors $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} + 2\hat{k}$.

Solution:- $\sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 1 & 3 & 2 \end{vmatrix} = (-2-9)\hat{i} - (4-3)\hat{j} + (6+1)\hat{k}$$

$$= -11\hat{i} - \hat{j} + 7\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-11)^2 + (-1)^2 + 7^2} = \sqrt{171} = 3\sqrt{19}$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\therefore \sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|} = \frac{3\sqrt{19}}{\sqrt{14}\sqrt{14}} = \frac{3}{14}\sqrt{19} \quad (\text{Ans}).$$

Example-4:-

Find the area of the parallelogram whose adjacent sides are represented by the vectors $(2\hat{i} + \hat{j} - 2\hat{k})$ and $\hat{i} - 3\hat{j} + 4\hat{k}$.

Solution:-

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (4-6)\hat{i} - (12+2)\hat{j} + (-9-1)\hat{k}$$

$$= -2\hat{i} - 14\hat{j} - 10\hat{k}$$

Required area = $|\vec{a} \times \vec{b}|$

$$= \sqrt{(-2)^2 + (-14)^2 + (-6)^2} \text{ sq units}$$

$$= \sqrt{300} \text{ sq units} = 10\sqrt{3} \text{ sq units}$$

Example-5 1-

Find the area of the parallelogram whose diagonals are represented by the vectors

$$\vec{d}_1 = (2\hat{i} - \hat{j} + \hat{k}) \text{ and } \vec{d}_2 = (3\hat{i} + 4\hat{j} - \hat{k})$$

Solution 1- Given that $\vec{d}_1 = 2\hat{i} - \hat{j} + \hat{k}$
 $\vec{d}_2 = 3\hat{i} + 4\hat{j} - \hat{k}$

vector area of parallelogram is $\frac{1}{2}(\vec{d}_1 \times \vec{d}_2)$

$$\text{Now } (\vec{d}_1 \times \vec{d}_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$$= (1-4)\hat{i} - (-2-3)\hat{j} + (8+3)\hat{k}$$

$$= -3\hat{i} + 5\hat{j} + 11\hat{k}$$

Required area = $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

$$= \frac{1}{2}\sqrt{(-3)^2 + (5)^2 + (11)^2} \text{ sq units}$$

$$= \frac{1}{2}\sqrt{155} \text{ sq units}$$

Example-6 1-

Using vector method, find the area of the triangle whose vertices are A(1,1,1), B(1,2,3) and C(2,3,1).

Solution 1-

position vector of A = $\hat{i} + \hat{j} + \hat{k}$

position vector of B = $\hat{i} + 2\hat{j} + 3\hat{k}$

position vector of C = $2\hat{i} + 3\hat{j} + \hat{k}$

$$\therefore \vec{AB} = (\text{P.V of B}) - (\text{P.V of A})$$

$$= (\hat{i} + 2\hat{j} + 3\hat{k}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{j} + 2\hat{k}$$

And $\vec{AC} = (\text{p.v of } C) - (\text{p.v of } A)$
 $= (2\hat{i} + 3\hat{j} + \hat{k}) - (\hat{i} + \hat{j} + \hat{k})$
 $= \hat{i} + 2\hat{j}$

Area of $\triangle ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

Now $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix}$

$= (0-4)\hat{i} + (2-0)\hat{j} + (0-1)\hat{k} = -4\hat{i} + 2\hat{j} - \hat{k}$

$\Rightarrow \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} |-4\hat{i} + 2\hat{j} - \hat{k}|$

$= \frac{1}{2} \sqrt{(-4)^2 + (2)^2 + (-1)^2}$

$= \frac{1}{2} \sqrt{16 + 4 + 1}$

$= \frac{1}{2} \sqrt{21} = \frac{\sqrt{21}}{2} \text{ sq units (Ans)}$

Assignment - 3

Q1. Find unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$, when

(i) $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$

(ii) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

Q2. Find the area of the parallelogram whose adjacent sides are represented by the vectors.

(i) $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -3\hat{i} - 2\hat{j} + \hat{k}$

(ii) $\vec{a} = 2\hat{i} + \hat{j}$ and $\vec{b} = \hat{j} - 3\hat{k}$

Q3. Find the area of the parallelogram whose diagonals are represented by the vectors.

(i) $\vec{d}_1 = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{d}_2 = \hat{i} - 3\hat{j} + 4\hat{k}$

(ii) $\vec{d}_1 = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{d}_2 = 3\hat{i} + 4\hat{j} - \hat{k}$

Q4. Using vector method find the area of $\triangle ABC$, whose vertices are $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$

Unit-2 (Limit & Continuity)

1. Limit

Let $y = f(x)$ be a function of x . If at $x = a$, $f(x)$ takes indeterminate form, then we consider the values of the function which is very near to a . If these values tends to a definite unique number as x tends to a , then the unique number, so obtained is called the limit of $f(x)$ at $x = a$ and we write it as $\lim_{x \rightarrow a} f(x) = L$ (2)

2. N.B.:- (Indeterminate forms)

If the value of $f(x)$ at $x = a$ comes out to one of the following forms:- $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 1^∞ , 0^0 and ∞^0 ;

then, it is called indeterminate forms.

3. Evaluation of limits (by direct substitution)

To find $\lim_{x \rightarrow a} f(x)$; we substitute $x = a$ in the function, if the value comes out to be a definite value, then it is the limit. i.e. $\lim_{x \rightarrow a} f(x) = f(a)$ provided it exists.

Example 1:- (1.5) - Evaluate

$$(i) \lim_{x \rightarrow 2} 2x + 3 = 2(2) + 3 = 4 + 3 = 7$$

$$(ii) \lim_{x \rightarrow 2} \frac{x^2 + 3}{x - 1} = \frac{2^2 + 3}{2 - 1} = \frac{7}{1} = 7$$

$$(iii) \lim_{x \rightarrow 3} \frac{x^3 + 3}{x} = \frac{3^3 + 3}{3} = \frac{27 + 3}{3} = \frac{30}{3} = 10$$

Rules of limit / (Algebra of limits)

(1) $\lim_{x \rightarrow a} k = k$

(2) $\lim_{x \rightarrow a} k f(x) = k \lim_{x \rightarrow a} f(x)$

(3) $\lim_{x \rightarrow a} f(x) \pm g(x) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$

(4) $\lim_{x \rightarrow a} f(x) \cdot g(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$

(5) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ (provided $\lim_{x \rightarrow a} g(x) \neq 0$)

(6) $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$

Example

(1) $\lim_{x \rightarrow 2} 5 = 5$

(2) $\lim_{x \rightarrow 3} (6x + 1) = \lim_{x \rightarrow 3} 6x + \lim_{x \rightarrow 3} 1 = 6 \lim_{x \rightarrow 3} x + 1 = 6 \times 3 + 1 = 18 + 1 = 19$

(3) $\lim_{x \rightarrow 5} 3x + x^2 + 5 = 3(5) + 5^2 + 5 = 15 + 25 + 5 = 45$

(4) $\lim_{x \rightarrow 2} e^x \ln x = e^2 \ln 2 = (2.718)^2 \times 0.693 = 4.0775$

(5) $\lim_{x \rightarrow 3} \frac{x^2}{\log x} = \frac{3^2}{\log 3} = \frac{9}{0.4771} = 18.86$

Evaluation of limit (Indeterminate forms)

These limit can be determined by some special rule for some standard function.

limit by factorisation :- $\left(\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1} \text{ form} \right)$

If the limit $f(x)$ attains Indeterminate form, $x \rightarrow a$ gbo

we know $x - a$ must be a factor of numerator and denominator which can be cancelled out.

$$\begin{aligned} \text{eg } \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{x \rightarrow a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \dots + a^{n-1})}{(x-a)} \\ &= \lim_{x \rightarrow a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1}) \\ &= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \dots + a^{n-1} \\ &= a^{n-1} + a^{n-1} + a^{n-1} + \dots \text{ } n \text{ terms} \end{aligned}$$

$$\Rightarrow \boxed{\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1}}$$

Example - 1

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} \quad \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 2} \frac{x^2 - 2^2}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \rightarrow 2} (x+2) = 2+2 = 4 \text{ (Ans)}$$

Example - 2

$$\lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3} \quad \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 3} \frac{x^3 - 3^3}{x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x^2 + 3x + 3^2)}{x - 3}$$

$$= 3^2 + 3 \cdot 3 + 3^2 = 3(3^2) = 27 \text{ (Ans)}$$

Example - 3

$$\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2} = \lim_{x \rightarrow 2} \frac{x^4 - 2^4}{x - 2} = 4(2)^{4-1} = 4(2)^3 = 4 \times 8 = 32$$

(Applying formulae)
 $\left[\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right]$

Example - 4

$$\lim_{x \rightarrow 1} \frac{x^n - 1}{x - 1} = n(1)^{n-1} = n \quad (\text{Ans})$$

(ii) Simplification - (Limit of polynomial and rational function)

(i) for $\lim_{x \rightarrow a} \frac{g(x)}{h(x)} = \frac{\lim_{x \rightarrow a} g(x)}{\lim_{x \rightarrow a} h(x)} = \frac{g(a)}{h(a)}$, for $\frac{g(a)}{h(a)}$

is a definite value.

(ii) If $\frac{g(a)}{h(a)}$ is not definite, then we have to factorise it.

example: $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^3 - 4x^2 + 4x} \left(\frac{0}{0} \right) = \lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{x(x^2 - 4x + 4)}$

$$\lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{x(x-2)^2} = \lim_{x \rightarrow 2} \frac{x+2}{x(x-2)} = \frac{2+2}{2(2-2)} = \frac{4}{0} = \infty$$

(iii) Limit at infinite - (Dividing highest power x^n or x^m)

$$\lim_{x \rightarrow \infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0}$$

(i) If $m \geq n$, divide x^m with each term, then put x as ∞ to find the value.

(ii) If $m < n$, divide x^n

(iii) If $m = n$, divide any x^m or x^n as both are equal.

Example - 1

$$\lim_{x \rightarrow \infty} \frac{4x^3 + 3x^2 + 1}{5x^2 + 2x + 1}$$

$$\text{Sol)} \quad \lim_{x \rightarrow \infty} \frac{\frac{4x^3}{x^3} + \frac{3x^2}{x^3} + \frac{1}{x^3}}{\frac{5x^2}{x^3} + \frac{2x}{x^3} + \frac{1}{x^3}} = \lim_{x \rightarrow \infty} \frac{4 + \frac{3}{x} + \frac{1}{x^3}}{\frac{5}{x} + \frac{2}{x^2} + \frac{1}{x^3}}$$

$$= \frac{4 + \frac{3}{\infty} + \frac{1}{\infty}}{\frac{5}{\infty} + \frac{2}{\infty} + \frac{1}{\infty}} = \frac{4 + 0 + 0}{0 + 0 + 0} = \frac{4}{0} = \infty \quad (\text{Ans})$$

Example - 2

$$\lim_{x \rightarrow \infty} \frac{x^2 + 2x + 1}{x^3 + 4x^2 + 3x + 2}$$

$$\text{Sol)} \quad \lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^3} + \frac{2x}{x^3} + \frac{1}{x^3}}{\frac{x^3}{x^3} + \frac{4x^2}{x^3} + \frac{3x}{x^3} + \frac{2}{x^3}} = \frac{\frac{1}{x} + \frac{2}{x^2} + \frac{1}{x^3}}{1 + \frac{4}{x} + \frac{3}{x^2} + \frac{2}{x^3}}$$

$$= \frac{\frac{1}{\infty} + \frac{2}{\infty} + \frac{1}{\infty}}{1 + \frac{4}{\infty} + \frac{3}{\infty} + \frac{2}{\infty}} = \frac{0}{1} = 0 \quad (\text{Ans})$$

Example - 3

$$\lim_{x \rightarrow \infty} \frac{4x^3 + x^2 + 2x + 1}{2x^3 + 3x + 2}$$

$$\text{Sol)} \quad \lim_{x \rightarrow \infty} \frac{\frac{4x^3}{x^3} + \frac{x^2}{x^3} + \frac{2x}{x^3} + \frac{1}{x^3}}{\frac{2x^3}{x^3} + \frac{3x}{x^3} + \frac{2}{x^3}} = \lim_{x \rightarrow \infty} \frac{4 + \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x^3}}{2 + \frac{3}{x} + \frac{2}{x^3}}$$

$$= \frac{4 + \frac{1}{\infty} + \frac{2}{\infty} + \frac{1}{\infty}}{2 + \frac{3}{\infty} + \frac{2}{\infty}} = \frac{4}{2} = 2 \quad (\text{Ans})$$

Assignment - 1 (A)

(1) $\lim_{x \rightarrow 3} x + 3$

(7) $\lim_{n \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$

(2) $\lim_{x \rightarrow 4} \frac{4x + 3}{x - 2}$

(8) $\lim_{x \rightarrow 1} \frac{x^{15} - 1}{x^{10} - 1}$

(3) $\lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x - 1}$

(9) $\lim_{x \rightarrow 2} \frac{x^3 - 4x^2 - 4x}{x^2 - 4}$

(4) $\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$

(10) $\lim_{x \rightarrow 2} \frac{x^3 - 9x^2}{x^2 - 5x + 6}$

(5) $\lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4}$

(11) $\lim_{x \rightarrow 1} \frac{x^2 - 4x + 3}{x^2 - 6x + 5}$

(6) $\lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$

(12) $\lim_{x \rightarrow 5} \frac{\sqrt{x} - \sqrt{5}}{x - 5}$

Assignment - 1 (B)

(1) $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

(4) $\lim_{h \rightarrow 0} \frac{(a+h)^3 - a^3}{h}$

(2) $\lim_{x \rightarrow b} \frac{x^8 - b^8}{x^3 - b^3}$

(5) .

(3) $\lim_{x \rightarrow 1} \frac{x^{1/m} - 1}{x^{1/n} - 1}$

Assignment - 1 (C)

(1) $\lim_{x \rightarrow \infty} \frac{3x^2 + 4x - 1}{2x^2 + x + 2}$

(5) $\lim_{n \rightarrow \infty} \frac{n!}{(n+1)! - n!}$

(2) $\lim_{n \rightarrow \infty} \frac{1 + 2 + 3 + \dots + n}{n^2}$

(6) .

(3) $\lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3}$

(4) $\lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4}$

Limit of Some (Standard function) (Standard forms)

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

proof $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = \lim_{x \rightarrow a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1})}{x-a}$

$$= \lim_{x \rightarrow a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1})$$

$$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \dots + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \dots \quad n \text{ terms}$$

$$= na^{n-1} \quad (\text{proved})$$

→

Example - 1

$$\lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3} = \lim_{x \rightarrow 3} \frac{x^3 - 3^3}{x - 3} = 3(3)^{3-1} = 3 \times 3^2 = 27 \quad (\text{Ans})$$

Example - 2

$$\lim_{x \rightarrow 3} \frac{x^3 - 27}{x^2 - 9} = \lim_{x \rightarrow 3} \frac{x^3 - 3^3}{(x-3)(x+3)} = \left(\lim_{x \rightarrow 3} \frac{x^3 - 3^3}{x-3} \right) \times \left(\lim_{x \rightarrow 3} \frac{1}{x+3} \right)$$

$$= 3 \times (3)^{3-1} \times \frac{1}{3+3} = 3 \times 3^2 \times \frac{1}{6^2} = \frac{9}{2} \quad (\text{Ans})$$

Example - 3

$$\lim_{x \rightarrow 25} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{5}}}{x - 25} \quad \left(\frac{0}{0} \right) \text{ form}$$

$$\neq \lim_{x \rightarrow 25} \frac{\frac{\sqrt{5} - \sqrt{x}}{\sqrt{5}x}}{x - 25} = \lim_{x \rightarrow 25} \frac{\sqrt{5} - \sqrt{x}}{x - 25} \times \frac{1}{\sqrt{5}x}$$

$$= \lim_{x \rightarrow 25} \left(\frac{\sqrt{5} - \sqrt{x}}{\sqrt{5}x} \times \frac{\sqrt{5} + \sqrt{x}}{x - 25} \times \frac{1}{\sqrt{5}x} \right)$$

$$= \lim_{x \rightarrow 25} \left(\frac{5 - x}{\sqrt{5}x} \times \frac{1}{x - 25} \times \frac{1}{(\sqrt{5} + \sqrt{x})\sqrt{5}x} \right)$$

$$\lim_{x \rightarrow 25} \frac{5-25}{\sqrt{5+25}} \times \frac{1}{25-25} \times \frac{1}{(\sqrt{5+25}) \cdot 25}$$

$$= \frac{-20}{0} = \infty \quad (\text{Ans})$$

Example - 3

$$\lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{x+h-x}$$

$$= \lim_{x+h \rightarrow x} \frac{(x+h)^3 - x^3}{x+h-x}$$

$$\text{Let } x+h = y$$

$$= \lim_{y \rightarrow x} \frac{y^3 - x^3}{y - x}$$

$$\left[\lim_{n \rightarrow a} \frac{x^n - a^n}{n - a} = na^{n-1} \right]$$

$$= 3(x)^{3-1}$$

$$= 3 \cdot x^2 \quad (\text{Ans})$$

11) (a) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$ (b) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \log_e e = 1$

proof

$$\begin{aligned}
 &\text{let } a^x - 1 = y \\
 &\Rightarrow a^x = y + 1 \\
 &\Rightarrow \log_e a^x = \log_e (1 + y) \\
 &\Rightarrow x \log_e a = \log_e (1 + y) \\
 &\Rightarrow x = \frac{\log_e (1 + y)}{\log_e a}
 \end{aligned}$$

$$\begin{aligned}
 &\text{As } x \rightarrow 0 \Rightarrow y = a^0 - 1 = 1 - 1 = 0 \\
 &\text{As } x \rightarrow 0 \Rightarrow y \rightarrow 0
 \end{aligned}$$

Now

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \lim_{y \rightarrow 0} \frac{y}{\frac{\log_e (1 + y)}{\log_e a}}$$

$$= \lim_{y \rightarrow 0} \log_e a \cdot \frac{y}{\log_e (1 + y)}$$

$$= \log_e a \left[\lim_{y \rightarrow 0} \frac{y}{\log_e (1 + y)} \right]$$

[$\log_e a$ is a constant]

$$= \log_e a \left[\lim_{y \rightarrow 0} \frac{1}{\frac{\log_e (1 + y)}{y}} \right]$$

$$= \log_e a \left[\frac{1}{\lim_{y \rightarrow 0} \frac{1}{y} \log_e (1 + y)} \right]$$

$$= \log_e a \frac{1}{\lim_{y \rightarrow 0} \log_e (1 + y) \cdot \frac{1}{y}} = \log_e a \frac{1}{\log_e \lim_{y \rightarrow 0} [(1 + y)^{1/y}]}$$

$$= \log a \frac{1}{\log e} = \log a \frac{1}{\log e^1}$$

$$\left[\lim_{y \rightarrow 0} (1+y)^y = e \right]$$

$$= \log e^1 \quad (\text{Ans})$$

$$\left[\log e^1 = 1 \right]$$

problems : Solved Examples

(1) $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$

Solution :- $\lim_{x \rightarrow 0} \frac{a^x - 1 - b^x + 1}{x} = \lim_{x \rightarrow 0} \frac{(a^x - 1) - (b^x - 1)}{x}$

$$= \lim_{x \rightarrow 0} \frac{a^x - 1}{x} - \lim_{x \rightarrow 0} \frac{b^x - 1}{x}$$

$$= \log a - \log b$$

$$= \log \left(\frac{a}{b} \right)$$

(2) $\lim_{x \rightarrow 0} \frac{3^x - 2^x}{x}$

Solution :- $\lim_{x \rightarrow 0} \frac{3^x - 2^x - 1 + 1}{x} = \lim_{x \rightarrow 0} \left[\frac{3^x - 1}{x} - \frac{2^x - 1}{x} \right]$

$$= \lim_{x \rightarrow 0} \left(\frac{3^x - 1}{x} \right) - \lim_{x \rightarrow 0} \left(\frac{2^x - 1}{x} \right)$$

$$= \log 3 - \log 2$$

$$= \log \left(\frac{3}{2} \right) \quad (\text{Ans})$$

(iii) $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

Proof :- $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = \lim_{x \rightarrow 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots}{x}$

$$= \lim_{x \rightarrow 0} \left(1 + \frac{x}{2} + \frac{x^2}{3} + \dots \right) = 1$$

OR $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = \lim_{x \rightarrow 0} (\log(1+x))^{\frac{1}{x}} = \log \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = \log e = 1$

Solved Examples

$$(1) \lim_{x \rightarrow 1} \frac{x-1}{\log_e x} = \frac{1}{\lim_{x \rightarrow 1} \frac{\log x}{x-1}} = \frac{1}{\lim_{x \rightarrow 1} \frac{\log x}{x-1}}$$

$$x-1 = y \Rightarrow x = 1+y$$

$$x \rightarrow 1 \Rightarrow y = 0$$

$$\lim_{y \rightarrow 0} \frac{\log(1+y)}{y} = \frac{1}{1} = 1 \quad (\text{Ans})$$

Assignment - 2

$$(1) \lim_{x \rightarrow 0} \frac{\log(1+x)}{3^x - 1}$$

$$(2) \lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{x}$$

$$(3) \lim_{x \rightarrow 0} \frac{8^x - 2^x}{x}$$

$$(4) \lim_{x \rightarrow 0} \frac{e^{3x} - e^{2x}}{x}$$

$$(5) \lim_{x \rightarrow 2} \frac{x-2}{\log_a(x-1)}$$

$$(6) \lim_{x \rightarrow 0} \frac{a^{mx} - 1}{b^{nx} - 1}, n \neq 0$$

$$(7) \lim_{x \rightarrow a} \frac{\log x - \log a}{x - a}$$

$$(8) \lim_{x \rightarrow 0} \frac{e^{bx} - e^{ax}}{x}$$

$$(iv) (a) \lim_{x \rightarrow 0} (1+x)^{1/x} = e$$

$$(b) \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Solved Example

$$(1) \lim_{x \rightarrow 0} (1+2x)^{1/x}$$

$$= \lim_{x \rightarrow 0} (1+2x)^{1/2x \cdot 2}$$

$$= \left[\lim_{x \rightarrow 0} (1+2x)^{1/2x} \right]^2$$

$$= e^2 \quad (\text{Ans})$$

$$(2) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{3x}\right)^x$$

$$= \lim_{x \rightarrow \infty} \left(1 + \frac{1}{3x}\right)^{\frac{3x}{3}}$$

$$= \left[\lim_{x \rightarrow \infty} \left(1 + \frac{1}{3x}\right)^{3x} \right]^{1/3}$$

$$= e^{1/3} \quad (\text{Ans})$$

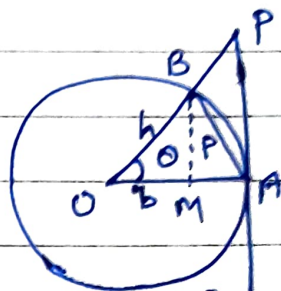
$$(a) \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$(b) \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

proof

$$(a) \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

consider a circle of radius r ,
 'O' be the centre of the circle.
 $\angle AOB = \theta$, measured in radians
 Tangent produced at P meets at A'.
 from figure, we can observe that



$$\sin \theta = \frac{P}{h} \Rightarrow P = h \sin \theta$$

Area of $\triangle OAB < \text{Area of sector } OAB < \text{Area of } \triangle OAP$
 $\frac{1}{2}(\text{base } OA \times \text{height } BM) < \frac{1}{2}(\text{radius } OA)^2 \theta < \frac{1}{2}(\text{base } OA \times \text{height } AP)$

$$= \frac{1}{2}(OA \times OB \sin \theta) < \frac{1}{2} r^2 \theta < \frac{1}{2} OA \cdot OA \tan \theta$$

$$= \frac{1}{2} r \times r \sin \theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan \theta$$

$$= \frac{1}{2} r^2 \sin \theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan \theta$$

$$\Rightarrow \sin \theta < \theta < \tan \theta$$

$$\text{In } \triangle OAP = \tan \theta = \frac{P}{b}$$

$$\Rightarrow \tan \theta = \frac{AP}{OA}$$

$$\Rightarrow AP = OA \tan \theta$$

$$(OA = OB = r)$$

$$\Rightarrow \frac{\sin \theta}{\sin \theta} < \frac{\theta}{\sin \theta} < \frac{\tan \theta}{\sin \theta}$$

$$\Rightarrow 1 < \frac{\theta}{\sin \theta} < \frac{\sin \theta}{\cos \theta \cdot \sin \theta}$$

$$\Rightarrow 1 > \frac{\sin \theta}{\theta} > \cos \theta$$

$$\Rightarrow \lim_{\theta \rightarrow 0} 1 > \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} > \lim_{\theta \rightarrow 0} \cos \theta$$

$$\Rightarrow 1 < \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} < 1 \Rightarrow \boxed{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1}$$

(b) ~~continuity~~

we have $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta}$

$$= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \times \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta} = 1 \times \frac{1}{1} = 1$$

Solved Example

Q1. $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$

$$= \lim_{x \rightarrow 0} \left(3 \times \frac{\sin 3x}{3x} \right) = 3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} = 3 \times (1) = 3$$

Q2. $\lim_{x \rightarrow 0} \frac{\sin 5x}{2x}$

$$= \lim_{x \rightarrow 0} \left(\frac{\sin 5x}{5x} \cdot 5x \right) \times \frac{1}{2x}$$

$$= \left(\lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \right) \times \frac{5x}{2x}$$

$$\boxed{\because \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = 1}$$

$$= \frac{5}{2} \times (1) = 5/2 \text{ (Ans).}$$

Q3. $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin ax}{ax} \cdot ax}{\frac{\sin bx}{bx} \cdot bx} = \frac{a}{b} \frac{\lim_{x \rightarrow 0} \left(\frac{\sin ax}{ax} \right)}{\lim_{x \rightarrow 0} \left(\frac{\sin bx}{bx} \right)} = \frac{a \cdot (1)}{b \cdot (1)} = \frac{a}{b}$$

Q4. $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} = 2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2$

$$= 2 \times (1)^2 = 2 \text{ (Ans)}$$

Assignment - 3

Q1. $\lim_{x \rightarrow 0} \frac{\sin 3x}{5x}$

Q2. $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x}$

[Hints $180^\circ = \pi^\circ$
 $1^\circ = \pi/180^\circ$
 $x^\circ = \frac{\pi x}{180}$]

Q3. $\lim_{x \rightarrow 0} \frac{\tan 4x}{\tan 6x}$

Q4. $\lim_{x \rightarrow 0} \frac{\tan 8x}{\sin 2x}$

Q5. $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$

Q6. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{2x^2}$

Q7. $\lim_{x \rightarrow \pi/2} \frac{\cos x}{\pi/2 - x}$

Q8. $\lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{1 - \sin x}$

Q9. $\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1$

Q10. $\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1$

Continuity

A real function $f(x)$ is said to be continuous at a point a if $\lim_{x \rightarrow a} f(x)$ exist and equals $f(a)$.

If $f(x)$ is not continuous at a point, it is said to be discontinuous at that point.

working? Rule

$f(x)$ is continuous at $x = 'a'$ if

(i) $f(a)$ has definite value at $x = a$

(ii) $\lim_{x \rightarrow a} f(x)$ exist.

(iii) $\lim_{x \rightarrow a} f(x) = f(a)$

N.B | Existence of limit

$\lim_{x \rightarrow a} f(x)$ exists iff $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$

both exist and are equal.

Evaluation of Left Hand Limit $\left[\lim_{x \rightarrow a^-} f(x) \right]$

" $x \rightarrow a^-$ " means that x is tending to a from the left hand side, i.e. x is a number less than a but very very close to ' a '.

Steps

> write $\lim_{x \rightarrow a^-} f(x)$

> put $x = a - h$ and replace $x \rightarrow a^-$ by $h \rightarrow 0$ to obtain $\lim_{h \rightarrow 0} f(a - h)$

> simplify $\lim_{h \rightarrow 0} f(a - h)$ by using the formula for the given function.

E.g.

Evaluate left hand limit of the function.

$$f(x) = \begin{cases} \frac{x-4}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases} \text{ at } x=4.$$

Solution 1:(L.H.L. of $f(x)$ at $x=4$)

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{h \rightarrow 0} f(4-h)$$

$$= \lim_{h \rightarrow 0} \frac{4-h-4}{4-h-4} \quad \left[\begin{array}{l} \text{Replacing } x \text{ by } 4-h \\ \text{in } f(x) = \frac{x-4}{x-4} \end{array} \right]$$

$$= \lim_{h \rightarrow 0} \frac{-h}{-h} = \lim_{h \rightarrow 0} \left(\frac{h}{-h} \right) \quad [\because h > 0]$$

$$= \lim_{h \rightarrow 0} (-1) = -1 \quad (\text{Ans.})$$

To Evaluate Right hand limit (RHL) $\left(\lim_{x \rightarrow a^+} f(x) \right)$
 $x \rightarrow a^+$ is equivalent to $x = a+h$ where $h > 0$,
 $\therefore$ where $h > 0$, always.

steps

> write $\lim_{x \rightarrow a^+} f(x)$

> put $x = a+h$ and replace $x \rightarrow a^+$ by $h \rightarrow 0$ to obtain $\lim_{h \rightarrow 0} f(a+h)$.

> simplify $\lim_{h \rightarrow 0} f(a+h)$ by using the formula.

e.g.

Evaluate the right hand limit of the function.

$$f(x) = \begin{cases} \frac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases} \quad \text{at } x=4.$$

solution(RHL of $f(x)$ at $x=4$)

$$\lim_{x \rightarrow 4^+} f(x)$$

$$= \lim_{h \rightarrow 0} f(4+h)$$

$$= \lim_{h \rightarrow 0} \frac{|4+h-4|}{4+h-4}$$

$$= \lim_{h \rightarrow 0} \frac{|h|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} (1) = 1 \quad (\text{Ans})$$

Solved ExamplesTest the continuity of the function $f(x)$ at the origin.

$$f(x) = \begin{cases} \frac{|x|}{x} & ; x \neq 0 \\ 1 & ; x = 0 \end{cases}$$

solution: we have.(LHL at $x=0$).

$$= \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h)$$

$$= \lim_{h \rightarrow 0} \frac{|-h|}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = \lim_{h \rightarrow 0} (-1) = -1.$$

and

(RHL at $x=0$)

$$= \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} f(h).$$

$$= \lim_{h \rightarrow 0} \frac{|h|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1.$$

$$\text{Thus, we have } \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x).$$

$\Rightarrow$ limit doesn't exist.

Hence, $f(x)$ is not continuous at the origin.

Q2. Discuss the continuity of the function $f(x)$ given by

$$f(x) = \begin{cases} 2-x, & x < 2 \\ 2+x, & x \geq 2. \end{cases}$$

Solution we have,

(LHL at $x=2$)

$$= \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2} 2-x$$

($\because f(x) = 2-x$ for $x < 2$)

$$= 2-2 = 0$$

(RHL at $x=2$)

$$= \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2} 2+x$$

$$= 2+2 = 4.$$

$$\therefore \lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$$

Hence $f(x)$ is not continuous at $x=2$, as limit doesn't exist.

N.B., $\therefore$ Hence $f(2) = 2+2 = 4$ (as $f(x) = 2+x$ at $x \geq 2$)
(Definite)

Assignment - 3

Q1. Examine the continuity of the function on $f(x)$ at the given point.

Q1. Examine the continuity of the function on $f(x)$ at the given point.

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases} \quad \text{at } x = 2$$

$$Q2. \therefore f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases} \quad \text{at } x = 1.$$

$$Q3. f(x) = \begin{cases} \frac{1 - \cos x}{x^2} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases} \quad \text{at } x = 0,$$

$$Q4. \therefore f(x) = \begin{cases} 2x - 1 & x < 0 \\ 2x + 1 & x \geq 0 \end{cases} \quad \text{at } x = 0,$$

(Q5) Determine the value of 'a' if the function $f(x)$ defined by $f(x) = \begin{cases} 2x - 1 & x < 2 \\ a & x = 2 \\ x + 1 & x > 2 \end{cases}$

is continuous at $x = 2$

Q2. For what value of k is the function

$$f(x) = \begin{cases} \frac{\sin 2x}{x} & , x \neq 0 \\ k & , x = 0 \end{cases}$$

continuous at $x = 0$.

Unit-III Differentiation

Introductory

Let $y = f(x)$ be a continuous function.
Then the value 'y' depends upon the value of x and it changes with a change in the value of 'x'.
So 'x' is called the independent variable, and 'y' is called the dependent variable.

Let Δx be a small change (+ve or negative) in 'x'.

Let Δy be the corresponding change in $y = f(x)$.
Then, the value of x changes from x to $x + \Delta x$ and the value of $f(x)$ changes from $f(x)$ to $f(x + \Delta x)$.

So the change in the value of f is $f(x + \Delta x) - f(x)$
or, $\boxed{\Delta y = f(x + \Delta x) - f(x)}$

Now, the average rate of change of y with respect to x is equal to $\frac{\Delta y}{\Delta x}$

As $\Delta x \rightarrow 0$, we find that Δy also tends to zero, therefore,

Instantaneous rate of change of y with respect to $x = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.

$$\Rightarrow \boxed{\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

measures the rate of change of $y = f(x)$ with respect to x which is called 'derivative of $f(x)$ w.r.t x '.

Geometrical meaning of derivative at a point

Consider the curve $y = f(x)$,
 let $f(x)$ be differentiable at $x = c$.
 Let $P(c, f(c))$ be a point on the curve.
 and $Q(x, f(x))$ be a neighbouring point on
 it. then,

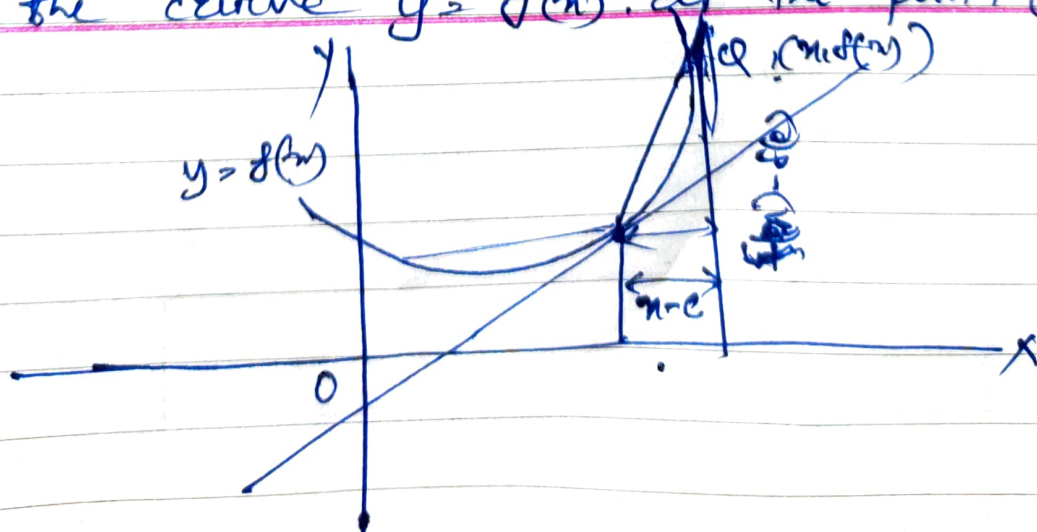
$$\text{slope of the chord } PQ = \frac{f(x) - f(c)}{x - c}$$

Taking limit as $Q \rightarrow P$, i.e. $x \rightarrow c$, we have,
 $\lim_{x \rightarrow c} (\text{slope of the chord } PQ) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$

As $Q \rightarrow P$, chord PQ becomes tangent
 at P . therefore, from (i), we have,
 Slope of the tangent at $P = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$

$$= \left[\frac{df(x)}{dx} \right]_{x=c} \text{ or } f'(c).$$

Thus the derivative of a function at
 a point $x = c$, is the slope of the tangent
 to the curve $y = f(x)$ at the point $(c, f(c))$



Differentiation from first principle

The derivative of a function is given by $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

The process of finding the derivative of a function by using the above definition is called the differentiation from first principle or by ab-initio method or by delta method.

N.B = $f'(x)$ can be also written as $\frac{dy}{dx} = Dy = y_1 = y'$

(we will find the derivative of some standard function by first principle)

Some Trigonometric formula to Recall.

$$(i) \sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \sin B$$

$$(ii) \cos(A \pm B) = \cos A \cdot \cos B \mp \sin A \cdot \sin B$$

$$(iii) \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$$

$$(iv) \tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A \cdot \cos B}$$

$$(v) \sin C + \sin D = 2 \sin \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$$

$$(vi) \sin C - \sin D = 2 \sin \frac{C-D}{2} \cdot \cos \frac{C+D}{2}$$

$$(vii) \cos C + \cos D = 2 \cos \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$$

$$(viii) \cos C - \cos D = -2 \sin \frac{C+D}{2} \cdot \sin \frac{C-D}{2}$$

$$(ix) 2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$(x) 2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$(xi) 2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$(xii) 2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

~~(xiii) $\sin A \pm \sin B$~~

Some limit formula recall

$$(i) \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$(ii) \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log a$$

$$(iii) \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = \frac{1}{\log a}$$

$$(iv) \lim_{x \rightarrow 0} \frac{\log_e(1+x)}{x} = 1$$

$$(v) \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$(vi) \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$$

$$(vii) \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

We will find out the following standard derivatives

$$(1) \frac{d}{dx} (x^n) = nx^{n-1}$$

$$(5) \frac{d}{dx} (\cos x) = -\sin x$$

$$(2) \frac{d}{dx} (a^x) = a^x \log_e a$$

$$(6) \frac{d}{dx} (\sin x) = \cos x$$

$$(3) \frac{d}{dx} (e^x) = e^x$$

$$(7) \frac{d}{dx} (\tan x) = \sec^2 x$$

$$(4) \frac{d}{dx} (\log_e x) = \frac{1}{x}$$

$$(8) \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$$

$$(10) \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$$

$$(9) \frac{d}{dx} (\sec x) = \sec x \tan x$$

$$(1) \frac{d(x^n)}{dx} = nx^{n-1}$$

proof Let $f(x) = x^n$, then $f(x+h) = (x+h)^n$

$$\therefore \frac{d(f(x))}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{(x+h) - x}$$

$$= \lim_{z \rightarrow x} \frac{z^n - x^n}{z - x} \quad \text{where } z = x+h \text{ and } z \rightarrow x \text{ as } h \rightarrow 0$$

$$= nx^{n-1}$$

Example 1:-

$$(i) \frac{d}{dx}(x^5) = 5x^{5-1} = 5x^4$$

$$(ii) \frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = \frac{-3}{x^4}$$

$$(iii) \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{1/2-1} = \frac{1}{2\sqrt{x}}$$

$$(iv) \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-1/2-1} = -\frac{1}{2}x^{-3/2}$$

$$(v) \frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$$

$$(vi) \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1x^{-1-1} = \frac{-1}{x^2}$$

(2) $\frac{d}{dx} (e^x) = e^x$

proof Let $f(x) = e^x$ then $f(x+h) = e^{x+h}$

$$\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx} (e^x) = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \rightarrow 0} e^x \left(\frac{e^h - 1}{h} \right) = e^x \lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right)$$

$$= e^x \cdot 1 = e^x \quad \left[\lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right) = 1 \right]$$

Hence $\frac{d}{dx} (e^x) = e^x$

(3) $\frac{d}{dx} (a^x) = a^x \log_e a$

proof Let $f(x) = a^x$ then $f(x+h) = a^{x+h}$

$$\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= a^x \lim_{h \rightarrow 0} \left(\frac{a^h - 1}{h} \right) = a^x \log_e a \quad \left[\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a \right]$$

$$\frac{d}{dx} (a^x) = a^x \log_e a$$

Example 1-

(i) $\frac{d}{dx} (5^x) = 5^x \log_e 5$

(ii) $\frac{d}{dx} (10^x) = 10^x \log_e 10$

(iii) $\frac{d}{dx} (e^{2x}) = \frac{d}{dx} (e^2)^x = (e^2)^x \log_e e^2 = e^{2x} 2 \log_e e = 2e^{2x}$

NB! δx is δt
 δx or $\delta t \approx 0$

$$\frac{d}{dx} (\sin x)$$

$$f(x) = \sin x$$

$$f(x + \delta x) = \sin(x + \delta x)$$

$$\frac{d}{dx} \sin x = \lim_{\delta x \rightarrow 0} \frac{\sin(x + \delta x) - \sin x}{\delta x} \quad (\text{by first principle})$$

$$= \lim_{\delta x \rightarrow 0} \frac{2 \sin\left(\frac{x + \delta x + x}{2}\right) \cdot \cos\left(\frac{x + \delta x - x}{2}\right)}{\delta x} \quad \left[\begin{array}{l} \sin C - \sin D \\ = 2 \sin \frac{C-D}{2} \cdot \cos \frac{C+D}{2} \end{array} \right]$$

$$= \lim_{\delta x \rightarrow 0} \frac{2 \sin\left(x + \frac{\delta x}{2}\right) \cdot \cos\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{2 \sin \frac{\delta x}{2}}{2x \cdot \frac{\delta x}{2}} \cdot \cos\left(\frac{\delta x}{2}\right)$$

$$= \lim_{\delta x \rightarrow 0} \left(\frac{\sin \delta x / 2}{\delta x / 2} \right) \times \cos\left(\frac{\delta x}{2}\right)$$

$$= 1 \times \lim_{\delta x \rightarrow 0} \cos\left(\frac{\delta x}{2}\right) = \cos\left(\frac{0}{2}\right) = 1$$

$$\Rightarrow \frac{d}{dx} \sin x = \cos x$$

Assignment 1 (prove by first principle / by definition)

① $\frac{d}{dx} (\cos x) = -\sin x$

② $\frac{d}{dx} (\tan x) = \sec^2 x$

③ $\frac{d}{dx} (\sec x) = \sec x \cdot \tan x$

Rules to find derivative,

① $\frac{d}{dx} c\{f(x)\} = c \frac{d}{dx} f(x)$

② $\frac{d}{dx} (f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

③ $\frac{d}{dx} [f(x) \cdot g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$

④ $\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2}$ $\frac{f}{g}$

Example: (find derivate of following function)

Q1 $y = 2x^3$

Ans $\frac{d}{dx} (2x^3) = 2 \frac{d}{dx} (x^3) = 2(3x^2) = 6x^2$

Q2 $y = x^3 - x^2 + 6$

Ans $\frac{d}{dx} (x^3 - x^2 + 6) = \frac{d}{dx} (x^3) - \frac{d}{dx} (x^2) + \frac{d}{dx} (6)$

$= 3x^2 - 2x + 0 = 3x^2 - 2x$ Ans

Q3 $y = x^2 \sin x$

Ans $\frac{dy}{dx} = \frac{d(x^2 \sin x)}{dx} = x^2 \frac{d}{dx} (\sin x) + \sin x \frac{d}{dx} (x^2)$

$= x^2 \cos x + \sin x (2x)$

$= x(2x \cos x + 2 \sin x)$ (Ans)

Q4 $y = \frac{\tan x}{e^x(1+x^2)}$

Ans $\frac{dy}{dx} = \frac{d(\tan x) / e^x(1+x^2)}{dx}$

$= \frac{e^x(1+x^2) \frac{d(\tan x)}{dx} - \tan x \frac{d}{dx} e^x(1+x^2)}{[e^x(1+x^2)]^2}$

$= \frac{e^x(1+x^2) \sec^2 x - \tan x [e^x \frac{d}{dx} (1+x^2) + (1+x^2) \frac{d(e^x)}{dx}]}{[e^x(1+x^2)]^2}$

$$= \frac{e^x \sec^2 x (1+x^2) - \tan x [e^x (2x) + (1+x^2)e^x]}{[e^x (1+x^2)]^2}$$

$$= \frac{e^x [(1+x^2)\sec^2 x - 2x\tan x - (1+x^2)\tan x]}{[e^x (1+x^2)]^2} \quad (\text{Ans})$$

Assignment 12 Find derivative of following function w.r.t x .

(i) $\frac{ax-b^2}{x}$ (ii) $\frac{\tan x}{\log x}$ (iii) $\frac{e^x}{1+x^2} - x \sin x$

Derivative of composite function by Chain Rule

Composite function: A function formed by composition of more than one function is called composite function.

example: 1) $\sin x^2$ (composite of $\sin x$ & x^2)

2) $a^{\ln x}$ (composite of a^x & $\ln x$)

3) $\sqrt{\sin \sqrt{x}}$ (composite of $\sqrt{x}$, $\sin x$ & $\sqrt{x}$)

(4) $\log(\sin x^2)$ (composite of $\log x$, $\sin x$, & x^2)

Chain Rule:-

If $y = f(t)$, t is a function of x defined by $t = g(x)$,

$$\text{then } \frac{dy}{dx} = \frac{df(t)}{dt} \times \frac{dt}{dx} = f'(t) \times g'(x)$$

In general: If $y = f(u)$
 $u = g(v)$
 $v = h(t)$

and so on,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dt} \cdots \frac{dt}{dx}$$

$\left\{ \begin{array}{l} t \text{ is a function of } x \\ \text{a independent variable} \end{array} \right.$

Example 1 find $\frac{dy}{dx}$ if $y = 5^{\sin x^2}$ at the end.

Soln

$$\frac{d}{dx} (5^{\sin x^2}) = \frac{d(5^{\sin x^2})}{d(\sin x^2)} \times \frac{d(\sin x^2)}{dx^2} \times \frac{dx^2}{dx}$$

$$= (5^{\sin x^2} \times \log 5) \times (\cos x^2) \times (2x)$$

$$\left\{ \begin{array}{l} \frac{d}{dx} a^x = a^x \log a \\ \frac{d}{dx} (\sin x) = \cos x \\ \frac{d}{dx} x^2 = 2x \end{array} \right.$$

PARTIAL DERIVATIVE

Page No.

compressing of two independent variables, namely, 'science' and 'Humanity'.

Partial Derivative:

A partial derivative of a function of several variables is its derivative with respect to one of those variables, with others held constant.

N.B. The symbol used to denote partial derivatives is ' ∂ ' called 'del'.

N.B. Several variable function:-

Suppose that 'A' denotes the area of a rectangle of side x and y units of length. Then $A = xy$

It is a function of two independent variables x & y . This fact is expressed by $A = f(x, y) = xy$.

Examples: pressure (p) : depends upon volume (V) & temp (T).
Output of a factory : depends upon independent (variable) factors : investment (capital), labour (input), work hours (time) etc.

Notation: $f(x, y, z, t, \dots)$

N.B. Function of two variable:

Let x, y , and z be three non-empty sets.
 f is a function of two variables defined from $x \times y$ to z . usually the notation is $z = f(x, y)$.
 $z \rightarrow$ dependent variable
 $x, y \rightarrow$ independent variable.

Geometrical Interpretation:-

A function $z = f(x, y)$ of two variables represents a surface in three dimension.

When we keep one variable constant say $y = b$, then the points on the surface intercepted by the plane $y = b$ determine a curve, $z = f(x, b)$.

The derivative of z w.r.t x at $x = a$ now gives us the slope.

Mathematics may not teach us how to add (one one minus hate). But it gives us energy reasons to hope that 'Energy problem has a solution'.

of this curve in the plane $y=b$ at the point $z=f(a,b)$

Hence $\frac{\partial z}{\partial x}$ means - slope of certain curve when y is kept constant. and so as we can define $\frac{\partial z}{\partial y}$.

N.B 1st order partial derivative $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ also written as f_x & f_y .

Example: $z = 2x^2y + xy^2 + 5x + 5y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

solution: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [2x^2y + xy^2 + 5x + 5y]$

$$= [2y \frac{\partial}{\partial x} (x^2) + y^2 \frac{\partial x}{\partial x} + 5 \frac{\partial x}{\partial x} + \frac{\partial (5y)}{\partial x}]$$

Taking y as a constant.

$$= 2y(2x) + y^2 + 5 + 0$$

$$= 4xy + y^2 + 5$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} [2x^2y + xy^2 + 5x + 5y]$$

$$= 2x^2 \frac{\partial y}{\partial y} + x \frac{\partial y^2}{\partial y} + \frac{\partial (5x)}{\partial y} + 5 \frac{\partial y}{\partial y}$$

Taking x as constant.

$$= 2x^2 + x(2y) + 0 + 5$$

$$= 2x^2 + 2xy + 5$$

solved problems:-

Q1. Given $z = \sin\left(\frac{x}{y}\right)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

solution: $\frac{\partial z}{\partial x} = \frac{\partial [\sin(\frac{x}{y})]}{\partial x} = \frac{\partial [\sin(\frac{x}{y})]}{\partial (\frac{x}{y})} \times \frac{\partial (\frac{x}{y})}{\partial x}$ [Chain Rule]

$$= \cos\left(\frac{x}{y}\right) \times \frac{1}{y} \frac{\partial x}{\partial x} \quad [\text{treating } y \text{ as constant}]$$

$$= \frac{1}{y} \cos\left(\frac{x}{y}\right) \quad (\text{Ans})$$

$$\frac{\partial z}{\partial y} = \frac{\partial [\sin(\frac{x}{y})]}{\partial y} = \frac{\partial [\sin(\frac{x}{y})]}{\partial (\frac{x}{y})} \times \frac{\partial (\frac{x}{y})}{\partial y}$$

$$= \cos\left(\frac{x}{y}\right) \times x \frac{\partial}{\partial y} \left(\frac{1}{y}\right) = \left[\cos\left(\frac{x}{y}\right)\right] \left(-\frac{x}{y^2}\right) \quad (\text{Ans})$$

pure mathematicians just love to try unsolved problems - They love a "challenge"

Q2. Find f_x and f_y if $f(x,y) = x^y + y^x$.

solution) $f_x = \frac{\partial f}{\partial x} = \frac{\partial (x^y + y^x)}{\partial x} = \frac{\partial (x^y)}{\partial x} + \frac{\partial (y^x)}{\partial x}$

$= yx^{y-1} + y^x \ln y$ (Ans)

$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [x^y + y^x]$

$= \frac{\partial (x^y)}{\partial y} + \frac{\partial (y^x)}{\partial y}$ (treating x as constant)

$= x^y \ln x + xy^{x-1}$ (Ans)

treating y as constant we can apply the formulae as $\rightarrow$
 $\frac{d(x^n)}{dx} = nx^{n-1}$
 $\frac{d(a^x)}{dx} = a^x \ln a$

Q3. Find f_x, f_y for $f(x,y) = \log(x^2 + y^2 - 2xy)$.

solution) $\log(x^2 + y^2 - 2xy) = \log(x-y)^2 = 2 \log(x-y)$

Now, $f_x = \frac{\partial}{\partial x} [2 \log(x-y)] = 2 \frac{\partial}{\partial x} \log(x-y)$

$= 2 \frac{\partial \log(x-y)}{\partial (x-y)} \times \frac{\partial (x-y)}{\partial x} = 2 \frac{1}{x-y} \times \left[\frac{\partial x}{\partial x} - \frac{\partial y}{\partial x} \right]$

$= \frac{2}{x-y} \times (1-0) = \frac{2}{x-y}$ (Ans) [taking y as constant]

$f_y = \frac{\partial}{\partial y} [2 \log(x-y)] = 2 \frac{\partial}{\partial y} \log(x-y) = 2 \frac{\partial \log(x-y)}{\partial (x-y)} \times \frac{\partial (x-y)}{\partial y}$

$= 2 \frac{1}{x-y} \times \frac{\partial (x)}{\partial y} - \frac{\partial y}{\partial y} = \frac{2}{x-y} \cdot (0-1) = -\frac{2}{x-y}$ (Ans)

Q4. If $z = f\left(\frac{y}{x}\right)$, show that $x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial y^2} = 0$

solution) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f\left(\frac{y}{x}\right)}{\partial x} \right) = \frac{\partial f\left(\frac{y}{x}\right)}{\partial \left(\frac{y}{x}\right)} \times \frac{\partial \left(\frac{y}{x}\right)}{\partial x} = f'\left(\frac{y}{x}\right) \times y \frac{\partial}{\partial x} \left(\frac{1}{x} \right)$

$= f'\left(\frac{y}{x}\right) \cdot y \left(-\frac{1}{x^2} \right) = -\frac{y}{x^2} f'\left(\frac{y}{x}\right)$ (Ans)

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f\left(\frac{y}{x}\right)}{\partial y} \right) = \frac{\partial f\left(\frac{y}{x}\right)}{\partial \left(\frac{y}{x}\right)} \times \frac{\partial \left(\frac{y}{x}\right)}{\partial y} = f'\left(\frac{y}{x}\right) \times \frac{1}{x} \frac{\partial y}{\partial y}$

$= \frac{1}{x} f'\left(\frac{y}{x}\right)$ (Ans)

Now, $x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial y^2} = x \left(-\frac{y}{x^2} \right) f'\left(\frac{y}{x}\right) + y \left(\frac{1}{x} \right) f'\left(\frac{y}{x}\right)$
 $= \left[-\frac{y}{x} + \frac{y}{x} \right] f'\left(\frac{y}{x}\right) = 0$ A.S.

"History" make men wise
"poets" witty.
the mathematician subtle; natural philosophy deep;

Assignment-1 (PD) moral grave; logic & rhetoric
able to contend.

1. Find dx, dy , where $f(x,y)$ is given as follows.

(i) $\frac{x^2y+y^2x}{x+y}$ (ii) $\sqrt{x^2+y^2}$ (iii) $\sin\left(\frac{x}{y}\right)$ (iv) e^{2x+3y}

(v) a^{xy} (vi) $e^{2x} \cos 3y$ (vii) $\tan(xy)$:

2nd order
order 3
order to
next section

2. If $f(x,y,z) = e^{xyz}$ then find $x dx + y dy + z dz$

3. If $U = (x^2+y^2+z^2)^{-1/2}$ show that,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

[Hints: take $x^2+y^2+z^2 = r$.

$$\Rightarrow u = r^{-1/2} \Rightarrow u = \frac{1}{\sqrt{r}} \Rightarrow u^2 = \frac{1}{r}$$

$$\Rightarrow \frac{\partial}{\partial x}(u^2) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right) \Rightarrow 2u \frac{\partial u}{\partial x} = -\frac{1}{r^2} \times \frac{\partial r}{\partial x} \Rightarrow (2u) \frac{\partial u}{\partial x} = -\frac{1}{r} (2x)$$

$$\Rightarrow u \frac{\partial u}{\partial x} = -\frac{x}{r} \Rightarrow \frac{1}{\sqrt{r}} \frac{\partial u}{\partial x} = -\frac{x}{r} \Rightarrow \boxed{\frac{\partial u}{\partial x} = -\frac{x}{\sqrt{r}}}$$

Again differentiate $\frac{\partial u}{\partial x}$ w.r.t. x to find $\frac{\partial^2 u}{\partial x^2}$
& similarly $\frac{\partial^2 u}{\partial y^2}$ & $\frac{\partial^2 u}{\partial z^2}$ in this manner.]

4. If $z = \sqrt{x} + \sqrt{y} + \sqrt{z}$, find $xU_x + yU_y + zU_z$.

5. If $z = xy f\left(\frac{y}{x}\right)$, show that $x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial y^2} + z \frac{\partial^2 z}{\partial z^2} = 3z$.

6. Given $f(u,v) = \frac{2u-3v}{u^2+v^2}$, find $f_u(2,1)$ and $f_v(2,1)$.

(Hints: find $f_u = \frac{\partial f}{\partial u}$ & $f_v = \frac{\partial f}{\partial v}$ by taking v & u as constant respectively, then put $u=2$ & $v=1$.)

Mathematics is not about number, equation, computation or algorithms, it is about understanding.
2nd order partial derivative:

for $z = f(x, y)$, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves function of two variables x and y .

Hence they also possess higher order partial derivative as following notations.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (f_x) = f_{xx}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} (f_y) = f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (f_y) = f_{xy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (f_x) = f_{yx}$$

Note: $f_{yx} = f_{xy}$ when partial derivatives are continuous.

Solved Problems:

Example:

Q. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ for $z = e^{2x} \cos 3y$.

Solution) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (f_x)$ & $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} (f_y)$

$$f_x = \frac{\partial}{\partial x} (e^{2x} \cos 3y) = \cos 3y \frac{\partial (e^{2x})}{\partial x} = \cos 3y (2e^{2x})$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} (\cos 3y (2e^{2x})) = 2 \cos 3y \frac{\partial (e^{2x})}{\partial x} = 2 \cos 3y (2e^{2x})$$

$$f_y = \frac{\partial}{\partial y} (e^{2x} \cos 3y) = e^{2x} \frac{\partial (\cos 3y)}{\partial y} = e^{2x} (-\sin 3y) \cdot 3$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} (-3e^{2x} \sin 3y) = -3e^{2x} \frac{\partial (\sin 3y)}{\partial y} = -3e^{2x} (\cos 3y) \cdot 3$$

Assignment 1) Solve Q. No. '3' of Assignment - 1 (P1).

2) Find f_{xx} , f_{yy} , f_{xy} , f_{yx} of Q. No. 1 (Assignment - 1 (P1))

Mathematics has beauty in its practice.

Homogeneous function:

Definition:- A function f is said to be homogeneous in x and y of degree ' n ' if $f(tx, ty) = t^n f(x, y)$, where ' t ' is any constant.

Note: To test where a function is homogeneous or not we have to replace x and y as tx & ty , then simplify the expression and taking common t^n (n is the least common power) we can conclude that form $t^n f(x, y)$.

Example-1: Test if $f(x, y) = 2xy^2 - 3x^2y$ is homogeneous or not.

Solution:

$$\begin{aligned} \text{Here } f(tx, ty) &= 2(tx)(ty)^2 - 3(tx)^2(ty) \\ &= 2(tx)t^2y^2 - 3t^2x^2(ty) \\ &= 2t^3xy^2 - 3t^3x^2y \\ &= t^3(2xy^2 - 3x^2y) \\ &= t^3 f(x, y). \end{aligned}$$

Hence f is a homogeneous of degree '3'.

Example-2: Test $\sin^{-1}\left(\frac{x}{y}\right)$ is homogeneous function and hence find its degree.

Solution:

$$\begin{aligned} f(x, y) &= \sin^{-1}\left(\frac{x}{y}\right) \\ f(tx, ty) &= \sin^{-1}\left(\frac{tx}{ty}\right) = \sin^{-1}\left(\frac{x}{y}\right) = f(x, y) \end{aligned}$$

$$\Rightarrow f(tx, ty) = t^0 f(x, y) \quad [t^0 = 1]$$

Hence $\sin^{-1}\left(\frac{x}{y}\right)$ is a homogeneous function of degree '0'.

Note: (1) A function is homogeneous if each term in the expression is of the same degree.

(2) If z is a homogeneous function of x and y of degree n , then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are also homogeneous of degree $n-1$.

(3) A homogeneous function of degree ' n ' can put in the form:

$$z = x^n \phi\left(\frac{y}{x}\right)$$

"The only way to learn mathematics is to do mathematics."

Euler's Theorem : c & t orders.

If z is a homogeneous function of degree n , then,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz.$$

Proof:

Given z is a homogeneous function of degree n .

We can put in the form as: $z = x^n \phi\left(\frac{y}{x}\right)$.

Now, $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [x^n \phi\left(\frac{y}{x}\right)]$.

$[x^n \& \phi\left(\frac{y}{x}\right)]$ both have variable x so we can apply

$$\left[\frac{d}{dx} [f(x)g(x)] = f'g + g'f \right]$$

$$= x^n \frac{\partial}{\partial x} \left[\phi\left(\frac{y}{x}\right) \right] + \phi\left(\frac{y}{x}\right) \frac{\partial (x^n)}{\partial x}$$

$$= x^n \phi'\left(\frac{y}{x}\right) \times \frac{\partial}{\partial x} \left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right) nx^{n-1}$$

$$= x^n \phi'\left(\frac{y}{x}\right) y \left(\frac{\partial}{\partial x} \frac{y}{x}\right) + nx^{n-1} \phi\left(\frac{y}{x}\right)$$

$$= x^n \phi'\left(\frac{y}{x}\right) y \left(-\frac{y}{x^2}\right) + nx^{n-1} \phi\left(\frac{y}{x}\right)$$

$$x \frac{\partial z}{\partial x} = -yx \cdot x^n \phi'\left(\frac{y}{x}\right) \frac{y}{x^2} + nx \cdot x^{n-1} \phi\left(\frac{y}{x}\right)$$

$$= -yx^{n-1} \phi'\left(\frac{y}{x}\right) + nx^n \phi\left(\frac{y}{x}\right). \text{ --- (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} [x^n \phi\left(\frac{y}{x}\right)]$$

$$= x^n \frac{\partial}{\partial y} \left[\phi\left(\frac{y}{x}\right) \right] \text{ [Taking } x \text{ as constant]}$$

$$= x^n \phi'\left(\frac{y}{x}\right) \frac{\partial}{\partial y} \left(\frac{y}{x}\right)$$

$$= x^n \phi'\left(\frac{y}{x}\right) \frac{1}{x} \frac{\partial y}{\partial y} = x^{n-1} \phi'\left(\frac{y}{x}\right).$$

$$y \frac{\partial z}{\partial y} = yx^{n-1} \phi'\left(\frac{y}{x}\right) \text{ --- (11)}$$

$$\therefore x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = -yx^{n-1} \phi'\left(\frac{y}{x}\right) + nx^n \phi\left(\frac{y}{x}\right) + yx^{n-1} \phi'\left(\frac{y}{x}\right) = nx^n \phi\left(\frac{y}{x}\right) = nz$$

Mathematics is the music of 'reason'

Postulates of Euler's theorem: (2nd order)

If z is a homogeneous function of x and y of degree n , then show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

proof: Given that $z = f(x, y)$ is homogeneous of degree n by Euler's theorem, we have

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \quad \text{--- (1)}$$

Differentiating eq (1) with respect to x ,

$$\frac{\partial}{\partial x} [x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}] = \frac{\partial}{\partial x} (nz)$$

$$\Rightarrow \left[\frac{\partial}{\partial x} \left(x \frac{\partial z}{\partial x} \right) \right] + \left[y \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \right] = n \frac{\partial z}{\partial x} \quad (\text{taking 'y' as constant})$$

$$\Rightarrow \left[x \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) + \frac{\partial z}{\partial x} \frac{\partial x}{\partial x} \right] + \left[y \frac{\partial^2 z}{\partial x \partial y} \right] = n \frac{\partial z}{\partial x}$$

$$\Rightarrow x \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y \frac{\partial^2 z}{\partial x \partial y} = n \frac{\partial z}{\partial x}$$

$$\Rightarrow x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} = (n-1) \frac{\partial z}{\partial x} \quad \text{--- (2)}$$

multiplying 'x' on both side

$$x^2 \frac{\partial^2 z}{\partial x^2} + xy \frac{\partial^2 z}{\partial x \partial y} = (n-1)x \frac{\partial z}{\partial x} \quad \text{--- (2)}$$

Differentiating eq (2) with respect to y ,

$$\frac{\partial}{\partial y} [x^2 \frac{\partial^2 z}{\partial x^2} + xy \frac{\partial^2 z}{\partial x \partial y}] = \frac{\partial}{\partial y} ((n-1)x \frac{\partial z}{\partial x})$$

$$\Rightarrow \left[x^2 \frac{\partial}{\partial y} \left(\frac{\partial^2 z}{\partial x^2} \right) \right] + \left[x \frac{\partial}{\partial y} \left(y \frac{\partial^2 z}{\partial x \partial y} \right) \right] = (n-1)x \frac{\partial^2 z}{\partial y \partial x}$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial y \partial x^2} + x \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial y \partial x} = (n-1)x \frac{\partial^2 z}{\partial y \partial x}$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial y \partial x^2} + x \frac{\partial^2 z}{\partial y^2} = (n-1)x \frac{\partial^2 z}{\partial y \partial x}$$

multiplying 'y' on both side

$$xy^2 \frac{\partial^2 z}{\partial y \partial x^2} + xy^2 \frac{\partial^2 z}{\partial y^2} = (n-1)xy \frac{\partial^2 z}{\partial y \partial x} \quad \text{--- (3)}$$

All you need

$$y = \frac{1}{x^2} \quad \frac{1}{x^2} = x^{-2}$$

$$x^2 + y^2 = 1 \quad \frac{1}{x^2} + y^2 = 1$$

$$y = 1 - 2x^2$$

$$x = -3 \sin y$$

Adding (2) and (3)

$$x^2 \frac{\partial^2 z}{\partial x^2} + xy \frac{\partial^2 z}{\partial x \partial y} + xy \frac{\partial^2 z}{\partial y \partial x} + y^2 \frac{\partial^2 z}{\partial y^2} = (n-1) [x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial y^2}]$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = (n-1) n z \quad [\text{proved}]$$

Example 8: (Application)

Q1. Verify Euler's theorem for $z = \frac{y}{x}$.

Ans: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left(\frac{y}{x} \right) = y \frac{\partial}{\partial x} \left(\frac{1}{x} \right) = y \left(-\frac{1}{x^2} \right) = -\frac{y}{x^2}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left(\frac{y}{x} \right) = \frac{1}{x} \left(\frac{\partial y}{\partial y} \right) = \frac{1}{x} (1) = \frac{1}{x}$$

Here $z = f(x, y) = \frac{y}{x}$

$$f(tx, ty) = \frac{ty}{tx} = \frac{y}{x} = f(x, y)$$

$$\Rightarrow f(tx, ty) = t^0 f(x, y),$$

$f(x, y) = \frac{y}{x}$ is homogeneous of degree '0'.

by statement of Euler's theorem $\cdot x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n z \quad (n=0)$

$$\Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = x \left(-\frac{y}{x^2} \right) + y \left(\frac{1}{x} \right) = -\frac{y}{x} + \frac{y}{x} = 0 \quad (\text{verified})$$

Q2. If $z = \sin^{-1} \left[\frac{x^2 + y^2}{x + y} \right]$, then prove that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \tan z$

Solution: Let $\frac{x^2 + y^2}{x + y} = u$ which is clearly a homogeneous function of degree '1'

Now $z = \sin^{-1} u$

$$\Rightarrow \sin z = u$$

$\Rightarrow [u = \sin z]$ is a homogeneous function of degree '1'

by Euler's theorem, $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u \quad (n=1)$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1 \cdot u$$

$$\Rightarrow x \frac{\partial (\sin z)}{\partial x} + y \frac{\partial (\sin z)}{\partial y} = 1 (\sin z) \Rightarrow x \cos z \frac{\partial z}{\partial x} + y \cos z \frac{\partial z}{\partial y} = \sin z$$

$$\Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{\sin z}{\cos z} = \tan z \quad (\text{proved})$$

Assignment:

- Q1. If $z = \frac{x-y}{x+y}$, then show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0$.
- Q2. Verify Euler's theorem for $u = x^2 \log\left(\frac{y}{x}\right)$.
- Q3. If $z = xy f\left(\frac{y}{x}\right)$, then show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$.
- Q4. If $z = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \sin 2z$.
- Q5. If $z = \cos^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = -\cot z$.
- Q6. If $z = \ln\left(\frac{x^2+y^2}{x+y}\right)$ then show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 1$.
- Q7. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x+y}}\right)$ show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u$.
- Q8. If $u = x^3 - 3xy^2$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.
- Q9. If $u = x \sin^{-1} \frac{y}{x}$ then show that $x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = 0$.
- Q10. If $u = \log r$, where $r^2 = (x-a)^2 + (y-b)^2$, then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.
- Q11. If $u = f\left(\frac{x^2+y^2}{xy}\right)$ prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$.
- Q12. Recall the geometrical meaning of partial derivative of 1st order two variable function. $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$.

“Thank you.”

Definite Integral:-

The definite integral is exactly defined as the limit and summation.

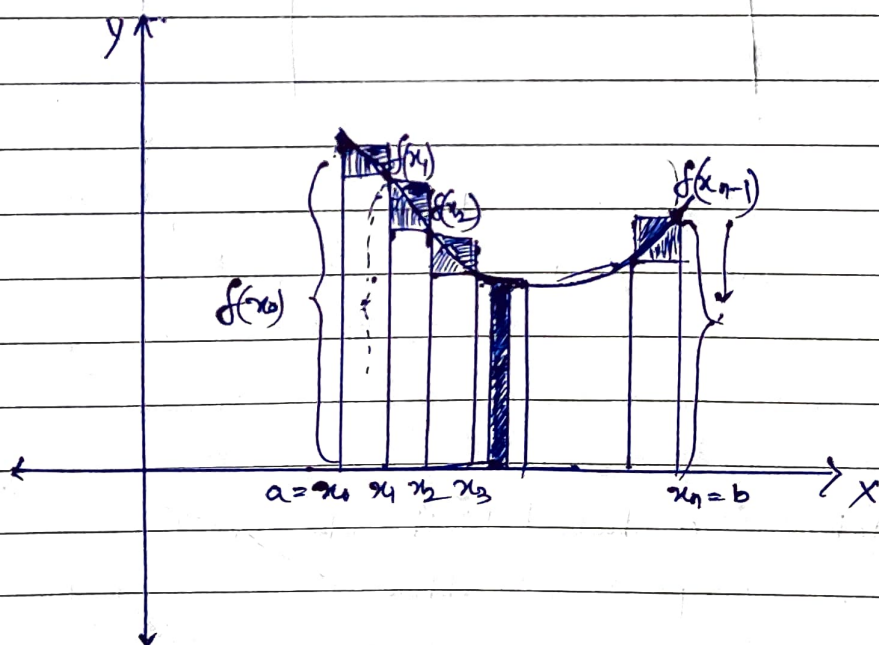
Derivation: (Definite integral as a limit of sum):

(Approximating the area under a curve):-

Geometric Concept: By dividing a region into many small shapes (rectangle) that have known area 'formula'; we can sum these area and obtain a reasonable estimate of the true area.

Consider Let $f(x)$ be a continuous and non negative function defined on the closed interval $[a, b]$.

The region is bounded by curve $f(x)$ above, the x -axis below, the line $x=a$ on the left and $x=b$ on the right.



Let the interval be divided into sub-intervals of length $\frac{b-a}{n}$ of equal width by the points $x_0, x_1, x_2, \dots, x_n$ such that $a = x_0 < x_1 < x_2 < x_3 < \dots < x_{n-1} < x_n = b$ and $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \dots = x_n - x_{n-1} = \frac{b-a}{n} = \Delta x$

Let's construct a rectangle with width Δx and height equal to $f(x_{i-1})$.

The area of the rectangle is 'length $\times$ breadth' = $f(x_{i-1}) \times \Delta x$.

Hence the total area enclosed by the curve, $y = f(x)$, $x = a$, $x = b$ and the x -axis is by taking sum of area of all rectangles as $\Rightarrow f(x_0) \Delta x + f(x_1) \Delta x + f(x_2) \Delta x + \dots + f(x_n) \Delta x$

Let the sum be $\boxed{S = \sum_{i=0}^n f(x_i) \Delta x} = \text{'Area' under the curve}$

(Note: Σ summation notation for 'n' numbers of terms to be added.)

But the actual area 'A' under the curve, $y = f(x)$ will be ^{nearly} accurate as $n \rightarrow \infty \Rightarrow \Delta x \rightarrow 0$

$$\text{Then } A = \lim_{n \rightarrow \infty} \sum_{i=0}^n f(x_i) \Delta x = \int_a^b f(x) dx$$

Thus the definite integral $\int_a^b f(x) dx$ represents the area under the curve $y = f(x)$ above the x -axis and between the ordinates $x = a$ and $x = b$.

Fundamental Theorem of Integral Calculus

(The relation between definite integral and indefinite integral)

statement:

If $f(x)$ is continuous in the interval $[a, b]$ and $\phi(x)$ is an anti-derivative of $f(x)$ [i.e. $\frac{d\phi(x)}{dx} = f(x)$]

$$\boxed{\int_a^b f(x) dx = \phi(b) - \phi(a)}$$

proof:- $\int f(x) dx = \phi(x) + k$ (where k is an arbitrary constant) $\Rightarrow \int_a^b f(x) dx = [\phi(x) + k]_a^b = \phi(b) + k - \phi(a) - k = \phi(b) - \phi(a)$

Example:-

$$\int_1^2 x^3 dx = \left| \frac{x^4}{4} \right|_1^2 = \frac{1}{4} |x^4|_1^2 = \frac{1}{4} |2^4 - 1| = \frac{1}{4} |16 - 1| = \frac{15}{4} \text{ (Ans)}$$

properties of definite integral. (Elementary Rules)

$$(1) \int_a^b [g(x) + h(x)] dx = \int_a^b g(x) dx + \int_a^b h(x) dx$$

amplifier $\int_0^{\pi/2} (3x^2 + 2x + \cos x) dx = \int_0^{\pi/2} 3x^2 dx + \int_0^{\pi/2} 2x dx + \int_0^{\pi/2} \cos x dx = \left[x^3 \right]_0^{\pi/2} + \left[x^2 \right]_0^{\pi/2} + \left[\sin x \right]_0^{\pi/2}$
 $= \frac{\pi^3}{8} + \frac{\pi^2}{4} + 1$

$$(2) \int_a^b \lambda g(x) dx = \lambda \int_a^b g(x) dx$$

$$(3) \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$(4) \int_a^b f(x) dx = \int_a^b f(y) dy = \int_a^b f(z) dz$$

$$(5) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \quad a < c < b$$

Some useful properties enlarging the scope of definite integrals.

$$(6) \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$(7) \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad (\text{if } f \text{ is even})$$

or 0 (if $f(x)$ is odd)

$$(8) \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \quad \text{if } f(2a-x) = f(x)$$

and 0 if $f(2a-x) = -f(x)$

(Total = 40 marks)

Assignments - I (D.I) (2 marks) $\times 8 =$ (16 marks)

Q1. $\int_{\pi/4}^{\pi/2} \tan^2 x \, dx$

5. $\int_1^2 e^{4x+1} \, dx$

2. $\int_{\pi/4}^{\pi/2} \sin 2x \cos x \, dx$

6. $\int_3^2 \frac{x+2}{3} \, dx$

3. $\int_{\pi/6}^{\pi/2} \frac{1+\cos 2x}{1-\cos 2x} \, dx$

7. $\int_0^2 x^2 e^{x^3} \, dx$

4. $\int_{\pi/4}^{\pi/2} \cot x \, dx$

8. $\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

(3 marks) $\times 4 =$ (12 marks)

9. $\int_0^1 \frac{dx}{\sqrt{e^{2x}-1}}$

(11) $\int_0^{3/2} [2x] \, dx$

10. $\int_0^4 \sqrt{x^2+9} \, dx$

(12) $\int_0^2 [x^2] \, dx$

4 marks $\times 3 =$ 12 (marks)

13. $\int_2^{2\pi} x \tan^{-1} x \, dx$

(15) $\int_0^{\pi/2} \frac{x}{(2x+1)(x+1)} \, dx$

(14) $\int_0^{\pi/2} \frac{\cos x \, dx}{(\sin x + 1)(\sin x + 2)}$

(16)

Assignment - 2 (Long Questions) 5x5=25 (marks)

(1) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

(2) $\int_0^{\pi/4} \log(1 + \tan x) dx$

(3) $\int_0^{\pi/2} \frac{dx}{1 + \cot x}$

(4) $\int_0^{\pi/2} \log(\sin x) dx$

(5) $\int_0^{\pi} \frac{x dx}{1 + \sin x}$

"You are never too old to set another goal or to dream a new dream." - C. S. Lewis

Differential Equations

An equation containing an independent variable, a dependent variable and the derivatives of the dependent variable is called a differential equation.

Example :- Each of the following equation is a differential equation :-

(i) $\frac{dy}{dx} + 5y = e^x$ (ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = \sin x$

(iii) $\frac{dy}{dx} = \frac{x^3 - y^3}{xy^2 - x^2y}$ (iv) $x^2 dx + y^2 dy = 0$

Order of Differential Equation :-

The order of the highest - Order derivative occurring in a differential equation is called the order of the differential equation.

Degree of a Differential Equation :-

The power of the highest - Order derivative occurring in a differential equation, after it is made free from radicals and fractions, is called the degree of the differential equation.

Example (i) Consider the equation $\left(\frac{dy}{dx}\right)^2 + 5y = \sin x$

The power of highest order and only one derivative term is 2.
so its order is = 1

degree is = 2

(ii) $x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + y^2 = 0$, order = 2
degree = 3

66 Do what you can with all you have, wherever you are. - Theodore Roosevelt

Some more example

$$(10) \quad \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

we have to make it free from radicals by squaring both the sides.

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

Clearly Order = 2
degree = 2,

Assignment -1

Determine the order and degree of each of the following equations.

$$1. \quad x^2 \left(\frac{d^2y}{dx^2}\right) + 2xy - 6x^3 = 0$$

$$2. \quad \left(\frac{d^2y}{dx^2}\right)^3 + 2\left(\frac{dy}{dx}\right)^4 + 9 = \sin x$$

$$3. \quad a \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^3\right\}^{3/2}$$

$$4. \quad x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$$

$$5. \quad x \left(\frac{dy}{dx}\right) + \frac{2}{\left(\frac{dy}{dx}\right)} = y^2 \quad (9) \quad \frac{dy}{dt} = \frac{y + \frac{dy}{dt}}{y + \frac{dy}{dt}}$$

$$6. \quad \sqrt{1-y^2}dx + y$$

$$7. \quad \left(\frac{dy}{dx}\right)^4 + 3y \frac{d^2y}{dx^2} = 0$$

$$(10) \quad \frac{d^2y}{dx^2} = 3y + \sqrt{\frac{d^2y}{dx^2}}$$

$$8. \quad \tan^{-1} \left(\frac{dy}{dx}\right) = x$$

$$(11) \quad e^{\frac{dy}{dx}} = x^2$$

$$9. \quad \ln \left(\frac{d^2y}{dx^2}\right) = y$$

$$(12) \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{3x}$$

66 "Fake it until you make it! Act as if you had all the confidence you required until it becomes your reality!"
- Brian Tracy

A Solution of a differential Equation

A function of the form $y = f(x) + C$ which satisfies a given differential equation is called its solution.

Solving Differential Equation with variable separation

1st order and 1st degree D.E.

(i) Differential equation of the type $\frac{dy}{dx} = f(x)$
method $\Rightarrow$ write $\frac{dy}{dx} = f(x)$

$$\Rightarrow dy = f(x) dx$$

$\Rightarrow$ Integrating both side,

$$\int dy = \int f(x) dx + C$$

$$\Rightarrow \boxed{y = \int f(x) dx + C}$$

Example 1 - solve $\frac{dy}{dx} = \frac{1}{x^2+1}$

$$\Rightarrow dy = \frac{1}{x^2+1} dx$$

$$\Rightarrow \int dy = \int \frac{1}{x^2+1} dx$$

$$\Rightarrow \boxed{y = \tan^{-1} x + C}$$

(ii) Differential equation of the type $\frac{dy}{dx} = g(y)$
method

write $\frac{dy}{dx} = g(y)$ form

$$\Rightarrow \frac{dy}{g(y)} = dx$$

$\Rightarrow$ Integrate $\int \frac{dy}{g(y)} = \int dx \Rightarrow \boxed{\int \frac{dy}{g(y)} = x + C}$

66 Dare to be free ; dare to go as far as your thought leads and dare to carry that out in your life - "Swami Vivekananda"

Example

$$\frac{dy}{dx} + y = 1$$

solution $\frac{dy}{dx} = 1 - y$

$$\Rightarrow \frac{dy}{1-y} = dx$$

$$\Rightarrow \int \frac{dy}{1-y} = \int dx$$

$$\Rightarrow \log|1-y| = x + C$$

$$\Rightarrow \log|1-y| = \log(e^{x+C})$$

$$\Rightarrow 1-y = e^{x+C}$$

$$\Rightarrow \boxed{y = 1 - e^{x+C}} \text{ (solution)}$$

Mr. B Always try to ^{express} ~~express~~ the solution as $y = f(x) + C$ form if possible.

(iv) Differential equation of the form $\frac{dy}{dx} = f(x)g(y)$ can be put in the form $\boxed{f(x) dx = g(y) dy}$

The solution is given by $\boxed{\int f(x) dx = \int g(y) dy + C}$

example $(x+1) \frac{dy}{dx} = 2xy$

$$\Rightarrow (x+1) dy = 2xy dx$$

$$\Rightarrow \cancel{(x+1)} \frac{dy}{y} = \frac{2x}{x+1} dx$$

Limitations live only in our minds. But if we use our imagination, our possibilities become limitless. - Jamie Paolinetti

$$\Rightarrow \int \frac{1}{y} dy = 2 \int \frac{x}{x+1} dx$$

$$\Rightarrow \int \frac{1}{y} dy = 2 \int \frac{x+1-1}{x+1} dx$$

$$\Rightarrow \int \frac{1}{y} dy = 2 \int \left(1 - \frac{1}{x+1}\right) dx$$

$$\Rightarrow \log y = 2[x - \log(x+1)] + C.$$

Assignment - 2

Solve

$$(i) \frac{dy}{dx} = x^2 + x - \frac{1}{x} \quad (iv) \frac{dy}{dx} = \log x$$

$$(ii) \frac{dy}{dx} + 2x = e^{3x} \quad (v) \frac{dy}{dx} = \tan^{-1} x$$

$$(iii) (x^2+1) \frac{dy}{dx} = 1$$

(2) Solve

$$(i) \frac{dx}{dy} = \sec y$$

$$(ii) \frac{dy}{dx} = \sin^2 y$$

$$(iii) \frac{dy}{dx} + y = 1$$

(3) Solve

$$(i) \sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$$

$$(ii) e^x \sqrt{1-y^2} dx + \frac{x}{y} dy = 0$$

$$(iii) \frac{dy}{dx} = e^{x+y}$$

If you are offered a seat on a rocket ship, don't ask what seat!
Just get on - Shreyl Sandberg

$$(iv) \frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1+x^2}} = 0$$

$$(v) \frac{dy}{dx} = (e^x + 1)y$$

$$(vi) x\sqrt{1-y^2} dy = (y+1)e^x dx$$

$$(vii) x \frac{dy}{dx} + \cot y = 0$$

$$(viii) \tan y dx + \sec^2 y \tan x dy = 0$$

$$(ix) \frac{dy}{dx} = y \sin x \quad y(0) = 1$$

$$(x) \tan^{-1} y = x + \frac{\pi^3}{3} + C$$

Linear Differential Equation

(i) The most general form of differential equation is $\frac{dy}{dx} + Py = Q$, where P is a constant, Q is a constant or a function of x .

(ii) The other common form of linear differential equation is $\frac{dx}{dy} + Px = Q$.

where P is a constant and Q is a constant, or a function of y .

Example

$$x \frac{dy}{dx} - y = x^2$$

which can be written as $\frac{dy}{dx} - \frac{1}{x} y = x$
as a linear differential equation.
where $P = -\frac{1}{x}$ and $Q = x$.

we can't help everyone, but everyone can help someone. -

- Ronald Reagan

solution of $\frac{dy}{dx} + Py = Q$.

steps

1) first write the differential equation in linear form and find out P & Q .

2) Next to find out $e^{\int P dx}$ which is known as the integrating factor, i.e. I.F.

3) multiply $e^{\int P dx}$ in the equation as

$$e^{\int P dx} \times \frac{dy}{dx} + Py e^{\int P dx} = Q e^{\int P dx}$$

$$\Rightarrow \frac{d}{dx} (e^{\int P dx} \times y) = Q e^{\int P dx}$$

4) Integrate with 'x'

$$\Rightarrow \int \frac{d}{dx} (e^{\int P dx} \times y) dx = \int Q e^{\int P dx} dx$$

$$\Rightarrow y \cdot e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

which is the required solution.

* M.B you can also omit the steps 3, 4. and directly find out the solution as

$y \times I.F. = \int (Q \cdot I.F.) dx + C$ which is general sol for all linear differential equation.

we can easily forgive a child who is afraid of the dark;
the real tragedy of life when men are afraid of the light.
- plato -

Solved Examples

Q1. solve $x \frac{dy}{dx} - y = x^2$

Solution the given differential equation may be written as $\frac{dy}{dx} - \frac{1}{x} y = x$

This is of the form $\frac{dy}{dx} + Py = Q$.

where $P = -\frac{1}{x}$ and $Q = x$

Thus, the given equation is linear.

$$I.f = e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

so the required solution is

$$\begin{aligned} y \times I.f &= \int (Q \times I.f) dx + C \\ \text{i.e. } y \times \frac{1}{x} &= \int \left(x \times \frac{1}{x}\right) dx + C \\ &= \int dx + C = x + C \end{aligned}$$

$$\therefore \frac{y}{x} = x + C \Rightarrow y = x^2 + Cx$$

Hence $y = x^2 + Cx$ is the required solution.

Q2. solve $(1+x^2) \frac{dy}{dx} + y = \tan^{-1} x$

Solution the given differential equation may be written as

$$\frac{dy}{dx} + \frac{1}{1+x^2} \cdot y = \frac{\tan^{-1} x}{1+x^2}$$

this is of the form $\frac{dy}{dx} + Py = Q$.

where $P = \frac{1}{1+x^2}$, $Q = \frac{\tan^{-1} x}{1+x^2}$.

Thus, the given equation is linear.

$$I.f = e^{\int P dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$$

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Be who you are and say what you feel,
because those who mind don't matter and
those who matter don't mind - Dr. Seuss

∴ the required solution is

$$y \times I f = \int (Q \times I f) dx + C$$

ie. $y \times e^{\tan^{-1} x} = \int \left\{ \frac{\tan^{-1} x}{1+x^2} \times e^{\tan^{-1} x} \right\} dx + C$

$$\begin{aligned} &= \int t e^t dt + C \text{ where } \tan^{-1} x = t \\ &= t e^t - \int 1 \cdot e^t dt + C \\ &= t e^t - e^t + C = e^t (t-1) + C \\ &= e^{\tan^{-1} x} (\tan^{-1} x - 1) + C \end{aligned}$$

Hence $y = \tan^{-1} x - 1 + C e^{-\tan^{-1} x}$ is required solution.

Example-3

Solve $\frac{dy}{dx} + (\sec x) y = \tan x$

Sol the given equation is of the form,

$$\frac{dy}{dx} + p y = Q \text{ where } p = \sec x \text{ and } Q = \tan x$$

$$I f = e^{\int p dx} = e^{\int \sec x dx} = e^{\log(\sec x + \tan x)} \\ = \sec x + \tan x$$

So the required solution is

$$\begin{aligned} y \times I f &= \int (Q \times I f) dx + C \\ \text{ie } y (\sec x + \tan x) &= \int \tan x (\sec x + \tan x) dx + C \\ &= \int \sec x \cdot \tan x dx + \int \tan^2 x dx + C \\ &= \sec x + \int (\sec^2 x - 1) dx + C \\ &= \sec x + \tan x - x + C \end{aligned}$$

So $y (\sec x + \tan x) = \sec x + \tan x - x + C$ is the required solution.