BHUBANANANDA ODISHA SCHOOL OF ENGINEERING, CUTTACK

DEPARTMENT OF CIVIL ENGINEERING

LECTURE NOTE ON: STRUCTURAL MECHANICS (TH-1) 3RD SEMESTER

PREPARED BY: SANGITA MOHANTY

GUEST FACULTY IN CIVIL ENGINEERING DEPT

BOSE, CUTTACK

Complex stresses

In real life, except for a few simple cases, the structural components and machine parts are not as simple, that they would be subjected only to one dimensional (uniaxial force) stress. Instead, components of machineries and structural elements of large and complex civil engineering structures are most likely to be subjected to complex three dimensional stress systems due to the system of forces acting on them. In such situations, the analysis and failure of the structural elements involve analysis of complex stresses.

Even when subjected to uniaxial stress, instead of failure at the plane, normal to the force, the component may fail due to yielding at a different plane due to induced shear stress exceeding the permissible shear stress of the material of the member.

Hence, it is not always the case that the plane normal or parallel to the force would experience the maximum stress. The element under direct tension or compression will experience shear stress and that under shear stress will experience normal stress albeit at different planes.

To obtain a complete picture of the stresses in a bar, we must consider the stresses acting on an “inclined” (as opposed to a “normal”) section through the bar.

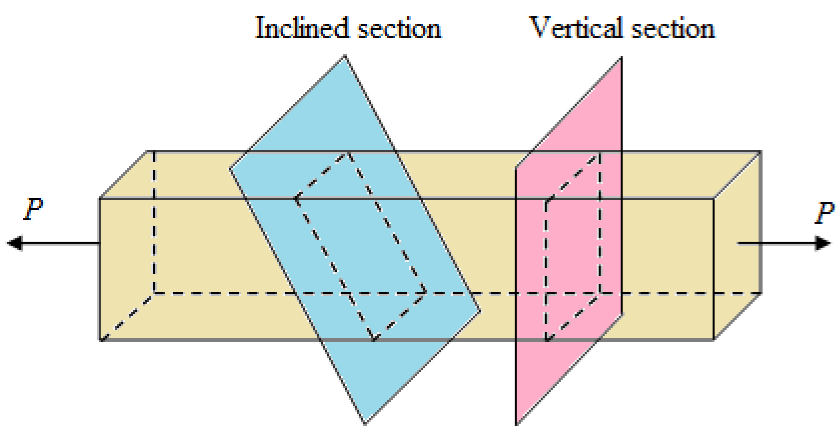


Fig. 1

Because the stresses are the same throughout the entire bar, the stresses on the sections are uniformly distributed.

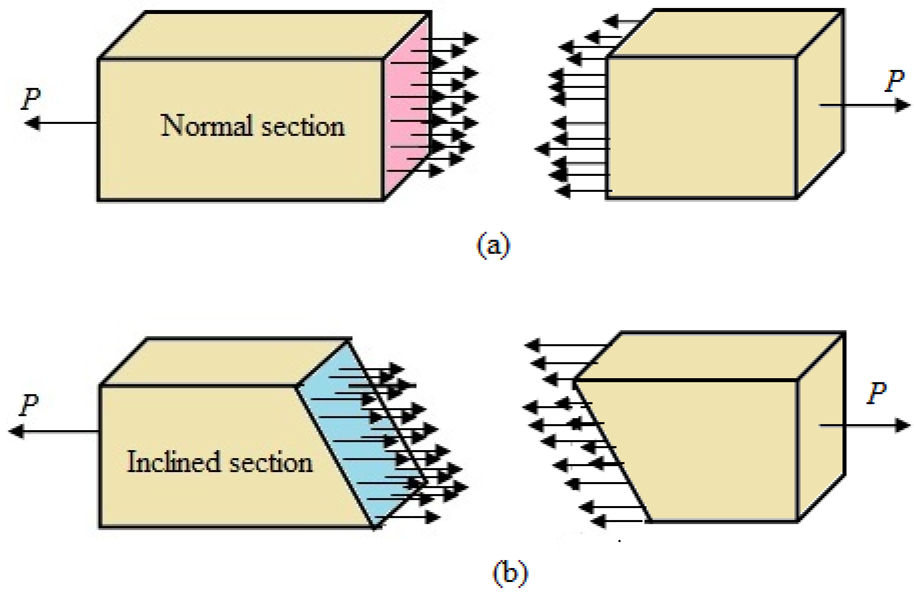


Fig. 2

Stresses in an oblique plane due to uniaxial normal stress:

In the previous section, we were computing the stresses on a plane when the force was either normal to the plane or tangential to it. In other words, the plane of interest was either normal to the force or parallel to it.

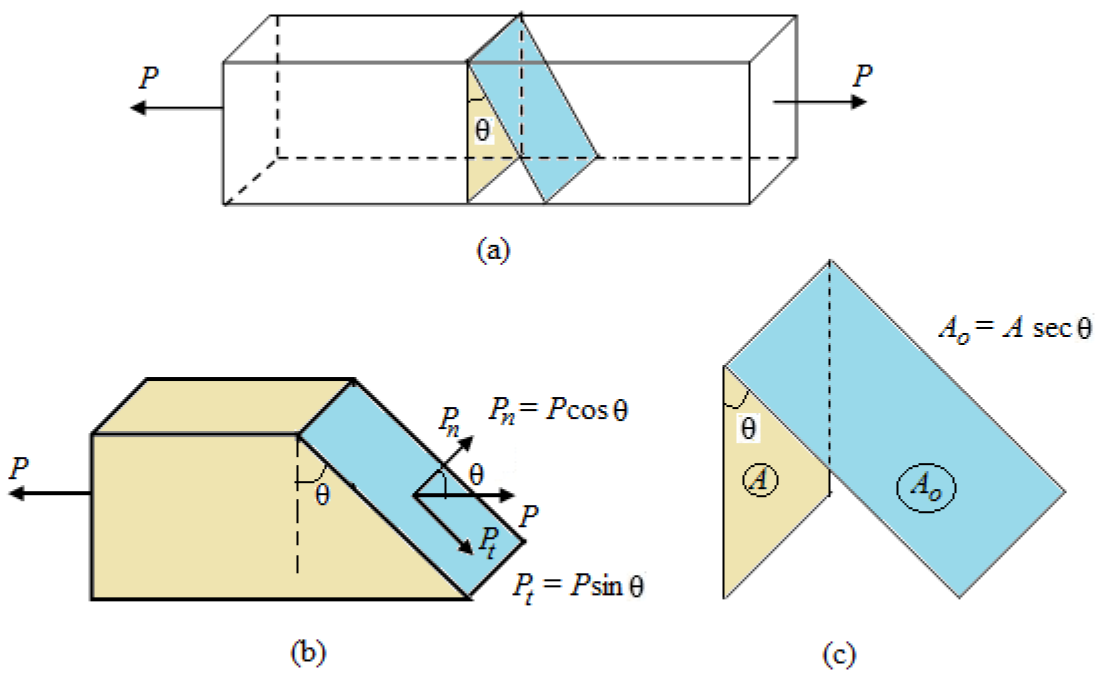


Fig. 3

In this section, the stresses on an oblique plane, i.e. plane making some angle (θ) with the vertical plane, will be dealt with. The normal stress on the vertical plane (perpendicular to the force) is $\sigma = P/A$, where P is the force and A is the area of cross-section of the plane.

In figure (b), however, there are two components of the force, one normal to the oblique plane (P_n) and the other tangential to the plane (P_t). Hence, the oblique plane will experience two stresses, one normal stress (σ_n) and the other tangential stress or shear stress (τ_n). The stress components can be calculated by dividing the respective forces with the area of the oblique plane.

Normal force perpendicular to the oblique plane, $P_n = P\cos\theta$

Shear force tangential to the oblique plane, $P_t = P\sin\theta$

Area of the oblique plane, $A_o = \dfrac{A}{\cos\theta} = A\sec\theta$

Normal stress,

$$\sigma_\theta = \frac{P_n}{A_o} = \frac{P\cos\theta}{A\sec\theta} = \frac{P\cos\theta.\cos\theta}{A}$$

$$= \sigma\cos^2\theta \qquad (1)$$

Tangential or shear stress,

$$\tau_\theta = \frac{P_t}{A_o} = \frac{P\sin\theta}{A\sec\theta} = \frac{P\sin\theta\cos\theta}{A}$$

$$= \frac{\sigma}{2}\sin 2\theta \qquad (2)$$

Thus, an oblique plane in a member under axial force will have two components of stresses on it, one normal to the plane while the other is tangential.

The stresses on the inclined plane, therefore, are not simply the resolutions of σ, perpendicular and tangential to that plane. The direct stress, σ_θ has a maximum value of σ, when $\theta = 0^0$ whilst the shear stress τ, has a maximum value of $\sigma/2$, when $\theta = 45^0$.

Thus, a material whose yield stress in shear is less than half that in tension or compression will yield initially in shear under the action of direct tensile or compressive forces.

Stresses in an oblique plane under pure shear stress system:

Consider a rectangular stressed element shown in Fig. 4 to which shear stresses (τ_{xy}) have been applied on the vertical sides so as to produce counterclockwise rotation. *Complementary shear stresses* of equal magnitude (τ_{yx}) but of opposite sense are then set up on the horizontal sides in order to prevent rotation of the element and to keep it in equilibrium.

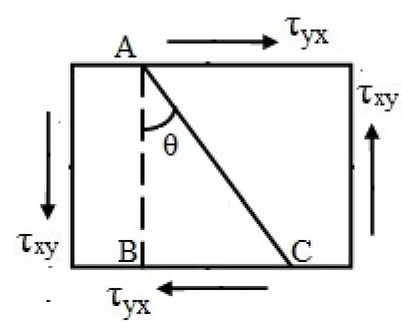


Fig. 4 Pure shear system

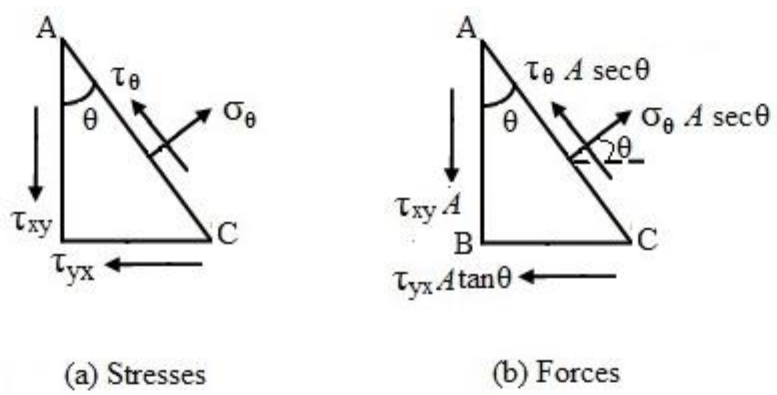


Fig. 5 Free body diagram of triangular prism

Consider the equilibrium of the triangular prism element in Fig. 5.

Resolving the forces along the direction normal to the plane AC, we have

$$\sigma_\theta . A\sec\theta = \tau_{xy} A\sin\theta + \tau_{yx} A\tan\theta\cos\theta$$

$$\sigma_\theta = \tau_{xy}\sin\theta\cos\theta + \tau_{yx}\sin\theta\cos\theta$$

$$= 2\tau_{xy} \sin\theta \cos\theta$$

$$= \tau_{xy} \sin 2\theta \qquad (3)$$

The maximum value of σ is τ_{xy} when $\theta = 45^0$.

Resolving the forces along the plane AC, we have

$$\tau_\theta . A\sec\theta + \tau_{yx} A\tan\theta \sin\theta = \tau_{xy} A\cos\theta$$

$$\tau_\theta . \sec\theta = \tau_{xy} \cos\theta - \tau_{yx} \tan\theta \sin\theta$$

$$\tau_\theta = \tau_{xy} \cos^2\theta - \tau_{yx} \sin^2\theta$$

$$= \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

$$= \tau_{xy} \cos 2\theta \qquad (4)$$

The maximum value of τ_θ, is τ_{xy} when $\theta = 0^0$ or 90^0 and it has a value of zero when $\theta = 45^0$, i.e. on the planes of maximum direct stress.

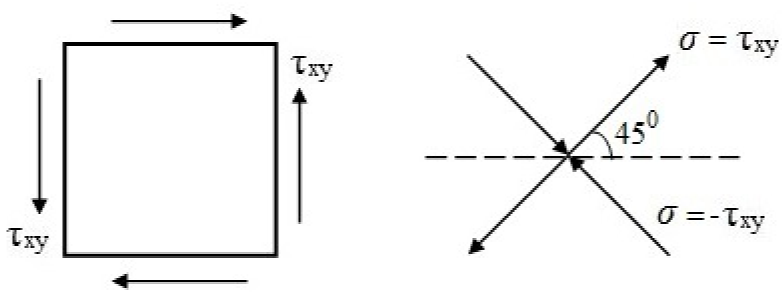


Fig. 6

Further consideration of eqn. (3) shows that the system of pure shear stresses produces an equivalent direct stress system as shown in Fig., one set compressive and one tensile, each at 45^0 to the original shear directions, and equal in magnitude to the applied shear.

Stresses in an oblique plane under biaxial normal stress system:

Consider the rectangular element as shown in Fig. 7 subjected to a system of two direct stresses, both tensile, at right angles, σ_x and σ_y.

For equilibrium of the portion *ABC*, resolving the forces perpendicular to *AC*,

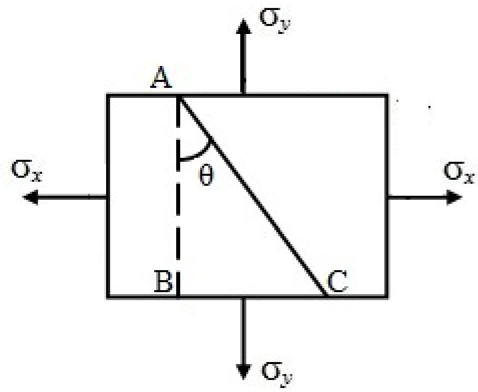


Fig. 7 Two -dimensioal normal stress system

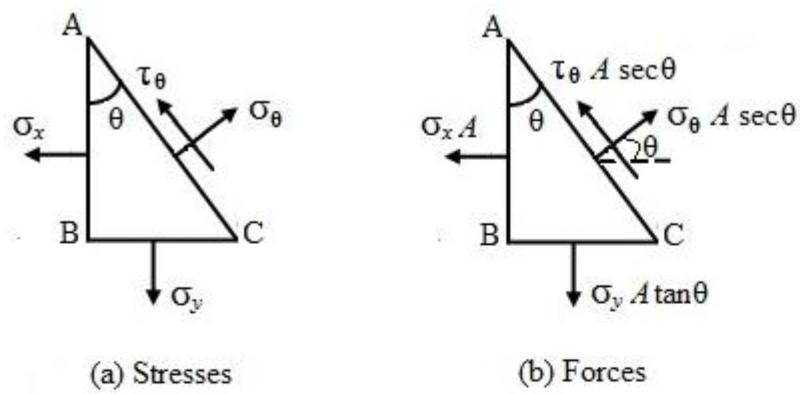


Fig. 8 Free body diagram of triangular prism

Consider the equilibrium of the triangular prism element in Fig. 8.

Resolving the forces along the direction normal to the plane AC, we have

$$\sigma_\theta . A\sec\theta = \sigma_x A.\cos\theta + \sigma_y A\tan\theta.\sin\theta$$

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta$$

$$= \frac{\sigma_x}{2}(1+\cos 2\theta) + \frac{\sigma_y}{2}(1-\cos 2\theta)$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \qquad (5)$$

Resolving the forces along the plane AC, we have

$$\tau_\theta . A\sec\theta + \sigma_x A\sin\theta = \sigma_y A\tan\theta.\cos\theta$$

$$\tau_\theta.\sec\theta = -\sigma_x \sin\theta + \sigma_y \tan\theta.\cos\theta$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta.\cos\theta$$

$$= -(\sigma_x - \sigma_y)\sin\theta\cos\theta$$

$$= -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta \qquad (6)$$

The maximum direct stress will equal σ_x or σ_y, whichever is the greater, when $\theta = 0$ or 90^0. The maximum shear stress occurs in *the plane* when $\theta = 45^0$.

$$\tau_{\max} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

Stresses in an oblique plane under general two-dimensional stress system:

Consider a rectangular prism of uniform cross-sectional area under bi-axial/ two-dimensional stress as shown in the Fig 9.

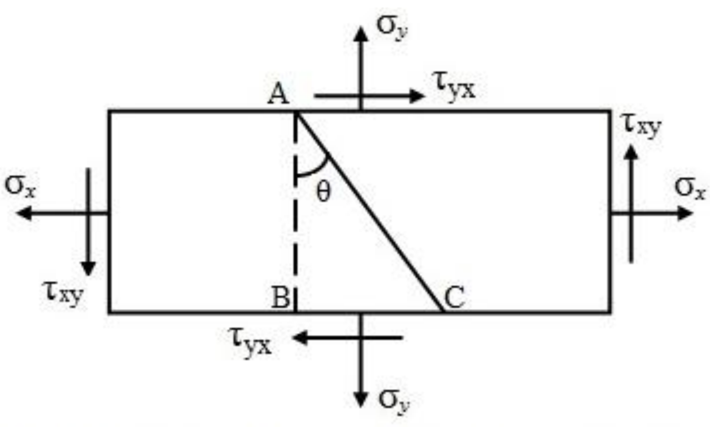


Fig. 9 Rectangular block under two-dimensional stress

Sign convention:

1. Angle of obliquity is measured in counter-clockwise direction with respect to vertical plane; the reference plane is taken as positive.
2. Tensile stress is taken as positive and compressive stress as negative.
3. Shear stress on vertical reference plane producing anticlockwise rotation is taken as positive.

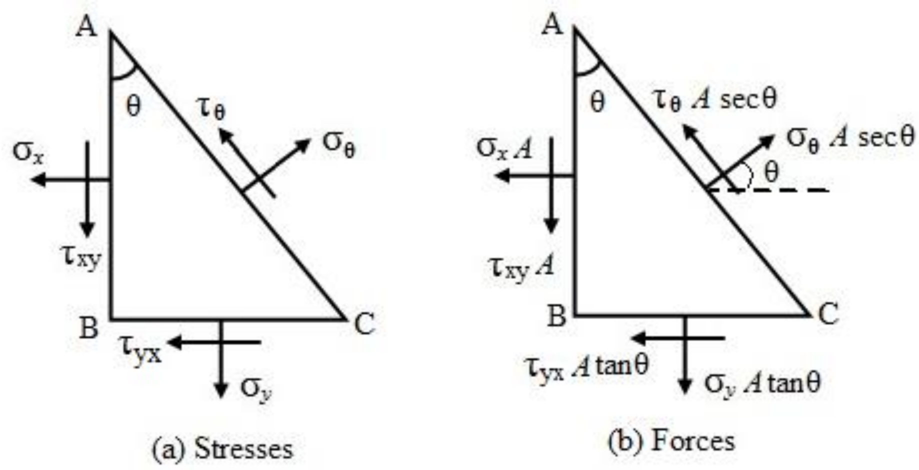


Fig.10 Free body diagram of triangular prism

Consider the equilibrium of the triangular prism element in Fig. 10.

Resolving the forces along the direction normal to the plane AC, we have

$$\sigma_\theta . A\sec\theta = \sigma_x A.\cos\theta + \tau_{xy} A\sin\theta + \sigma_y A\tan\theta.\sin\theta + \tau_{yx} A\tan\theta\cos\theta$$

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin\theta\cos\theta + \tau_{yx}\sin\theta\cos\theta$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$= \frac{\sigma_x}{2}(1+\cos 2\theta) + \frac{\sigma_y}{2}(1-\cos 2\theta) + \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (7)$$

Resolving the forces along the plane AC, we have

$$\tau_\theta . A\sec\theta + \sigma_x A\sin\theta + \tau_{yx} A\tan\theta\sin\theta = \sigma_y A\tan\theta.\cos\theta + \tau_{xy} A\cos\theta$$

$$\tau_\theta .\sec\theta = -\sigma_x \sin\theta - \tau_{yx}\tan\theta\sin\theta + \sigma_y \tan\theta.\cos\theta + \tau_{xy}\cos\theta$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta - \tau_{yx}\sin^2\theta + \sigma_y \sin\theta.\cos\theta + \tau_{xy}\cos^2\theta$$

$$= -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (8)$$

The maximum and minimum normal stresses (σ_1 and σ_2) are known as the **principal stresses**. The *maximum* and *minimum* normal stresses (σ_1 and σ_2) which occur on any plane in the member can now be determined as follows:

For σ_θ to be a maximum or minimum, $\frac{d\sigma_\theta}{d\theta} = 0$

We have $$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

Since we are dealing with maximum and minimum normal stresses (principal stress), let us denote this angle as θ_p. The above equation can be written as,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)} \qquad (9)$$

There are two values of $2\theta_p$ in the range 0-360°, with values differing by 180°. There are two values of θ_p in the range 0-180°, with values differing by 90°. So, the planes on which the maximum and minimum normal stresses act are mutually perpendicular and are known as ***principal planes***.

We can now solve for the principal stresses by substituting for θ_p in the normal stress equation for σ_θ.

From triangle I and II of Fig. 11,

$$\sin 2\theta_p = \pm\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\theta_p = \pm\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

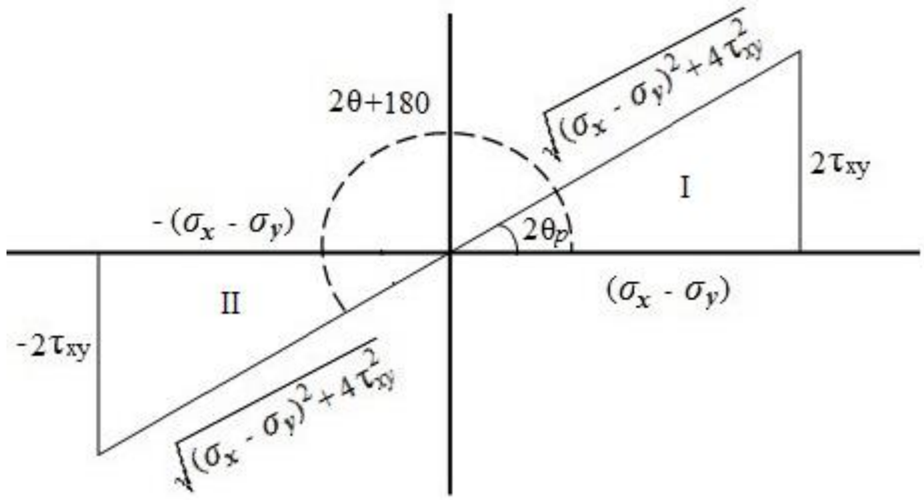


Fig. 11

Substituting the values of θ_p in (1), the maximum and minimum normal stresses are given by

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{(\sigma_x - \sigma_y)}{2} \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \pm \tau_{xy} \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \qquad (10)$$

We have derived the maximum and minimum values of the normal stresses denoted as σ_1 (maximum) and σ_2 (minimum).

To find out which principal stress is associated with which principal angle, we could use the equations for sin θ_p and cos θ_p or for σ_θ.

Similarly substituting the value of θ_p in eqn. (8),

$$\tau_{\theta_p} = \mp \frac{(\sigma_x - \sigma_y)}{2} \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \pm \tau_{xy} \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\tau_{\theta_p} = 0$$

Hence, the shear stresses are zero on the principal planes.

Now let us find out the maximum value of the shear stress. Differentiate eqn. (8) with respect to θ and equate it to zero. We get,

$$\frac{d\tau_\theta}{d\theta} = -\frac{(\sigma_x - \sigma_y)}{2}.2\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\tau_{xy}\sin 2\theta = -\frac{(\sigma_x - \sigma_y)}{2}.\cos 2\theta$$

Since we are dealing with shear, let us denote this angle as θ_s. The above equation can be written as,

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}} \qquad (11)$$

There are two values of $2\theta_s$ in the range 0-360°, with values differing by 180°. There are two values of θ_s in the range 0-180°, with values differing by 90°. So, the planes on which the maximum shear stresses act are mutually perpendicular.

Because shear stresses on perpendicular planes have equal magnitudes, the maximum positive and negative shear stresses differ only in sign.

Comparing the equation (4) for θ_s with that the equation (5) for θ_p, it is observed that both are reciprocal of each other. So we can write,

$$\tan 2\theta_s = -\frac{1}{\tan 2\theta_p} = -\cot 2\theta_p$$

$$\tan 2\theta_s = \tan(90 + 2\theta_p)$$

$$\theta_s = 45 + \theta_p$$

So, the planes of maximum shear stress (θ_s) occur at 45° to the principal planes (θ_p). We have therefore derived maximum & minimum values of principal stresses, their angles, maximum values of shear stress and its orientation with respect to principal planes.

Further, from triangle I and II of Fig. 12,

$$\sin 2\theta_p = \mp \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\theta_p = \pm \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

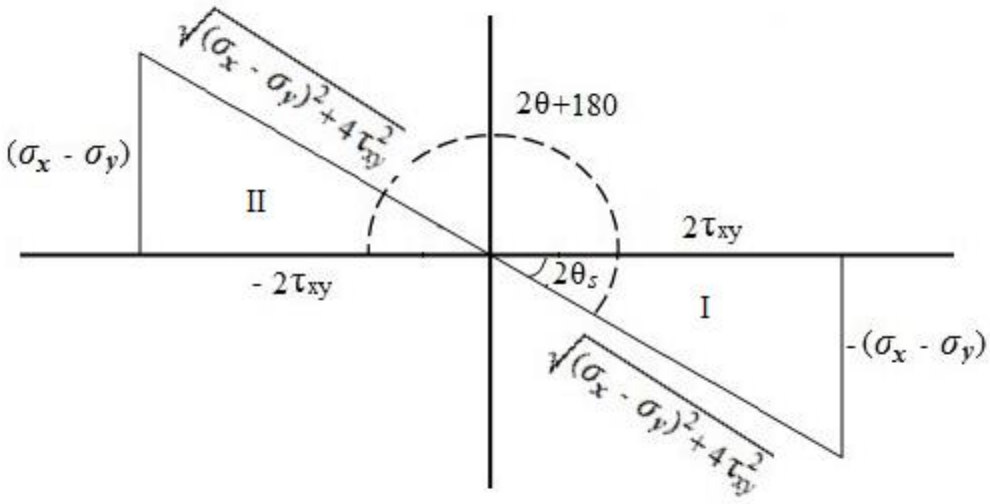


Fig. 12

Substituting the values of θ_s in (2), the maximum and minimum shear stresses are given by

$$\tau_{\max} = \pm \frac{(\sigma_x - \sigma_y)}{2} \cdot \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \pm \tau_{xy} \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\tau_{\max} = \pm . \frac{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}{2\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} = \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$\tau_{\max} = \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

In a structural member under biaxial complex stress system, there exist two mutually perpendicular planes on which the normal stress is either maximum or minimum. These planes are known as ***principal planes***. The shear stress on these planes is zero.

A little observation will show that, $\tau_{\max} = \pm \frac{1}{2}(\sigma_1 - \sigma_2)$, where σ_1 and σ_2 are principal stresses.

Further, it can be seen that the sum of the principal stresses is same as the sum of the normal stresses in any two mutually perpendicular directions.

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$$

Transformation of stress coordinates:

There exists only one intrinsic/fundamental state of stress at a point in a stressed body akin to the position of a point on a plane. As the coordinates of a point (x, y) on a plane changes with the orientation of coordinate axes of reference, so does the stress coordinates (σ, τ) of a point in a stressed body (with respect to the plane of reference) with the orientation of its plane of consideration.

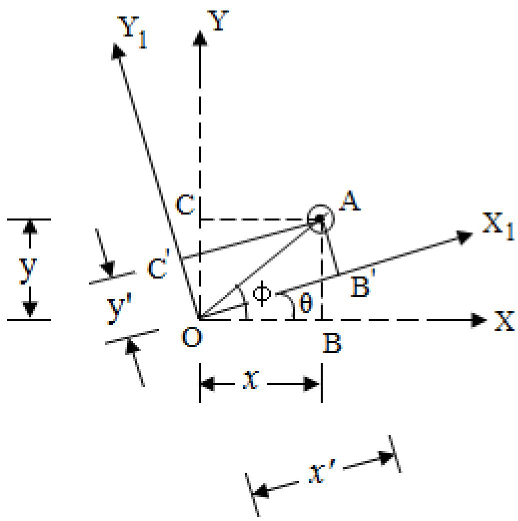


Fig. 13

'A' is a point on a plane with coordinates axes OX and OY as shown in Fig. 13. The coordinates of A is (x, y). If the orientation of axes of reference OX and OY is changed by an angle of θ in counterclockwise direction, then the position is not going to be changed. Instead its coordinates (x_1, y_1) in respect of changed coordinate axes will change.

$$d = \text{disance of } A \text{ from the origin } O \text{ of the plane}$$

$$\phi = \text{angular distance of } A \text{ from the origin } O \text{ of the plane}$$

$$x = d\cos\phi,\ y = d\sin\phi$$

$$x_1 = d\cos(\phi - \theta), y_1 = d\sin(\phi - \theta)$$

It is evident that only the coordinates of the point and not its inherent position depends on the orientation of the axes of reference. Similarly, regardless of the orientation of the element used to portray the state of stress, the intrinsic state of stress remains the same. In other words the stress coordinates (σ, τ) and not the inherent state of stress of a point change.

Member subjected to principal stresses:

Mohr's circle:

Mohr's circle is a graphical representation of stress transformation equations. The equations of stress transformation describe a circle if normal stress and shear stress are represented as abscissa and ordinate respectively. Each point on the circumference of Mohr's circle represents a plane through the centre of the circle and the coordinates (σ, τ) of the point represents the normal stress (σ) and shear stress (τ) on the given plane.

Mohr's circle can be drawn from a given state of stress at a point in a structural member. Consider a stress element representing the state of stress at a point as shown in the Fig. 14.

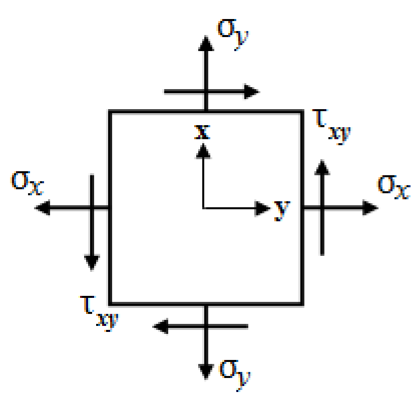


Fig. 14

Stress transformation equations for normal and tangential components on a plane are given by

Normal stress on the plane, $$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (12)$$

Shear stress on the plane, $$\tau_\theta = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (13)$$

Rearranging the equation (12), we have

$$\sigma_\theta - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (14)$$

Squaring both sides of equation (13) and (14) and adding them together, we have

$$\left(\sigma_\theta - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\theta^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$\left(\sigma_\theta - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\theta^2 = \left(\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\right)^2$$

This is the equation of a circle with centre $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$ and radius $R = \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ and this circle is known as Mohr's circle named after the German Civil Engineer Otto Mohr (1835-1918). It provides a simple and clear picture of an otherwise complicated analysis.

Procedure for drawing Mohr's circle:

1. Draw coordinates axes in Cartesian coordinate system with *O* as origin, normal stress (σ) as abscissa (positive to the right) and shear stress (τ) as ordinate (positive downward).
2. Locate the centre C of the circle at the point having coordinates $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$.
3. Locate point *A*, representing the state of stress on the vertical plane, i.e., face *x* of the element by plotting its coordinates σ_x and τ. Point *A* on the circle corresponds to $\theta = 0^0$ and represents the vertical plane.
4. Locate point *B*, representing the state of stress on the horizontal plane, i.e., face y of the element by plotting its coordinates σ_y and -τ. Point *A* on the circle corresponds to $\theta = 90^0$ and represents the horizontal plane.
5. Join *AB* so as to intersect the normal stress axis at *C*.
6. With the point *C* as the centre and *CA* (= *CB*) as radius, draw Mohr's circle through points *A* and *B*. This is the required Mohr's circle which has radius *R*.

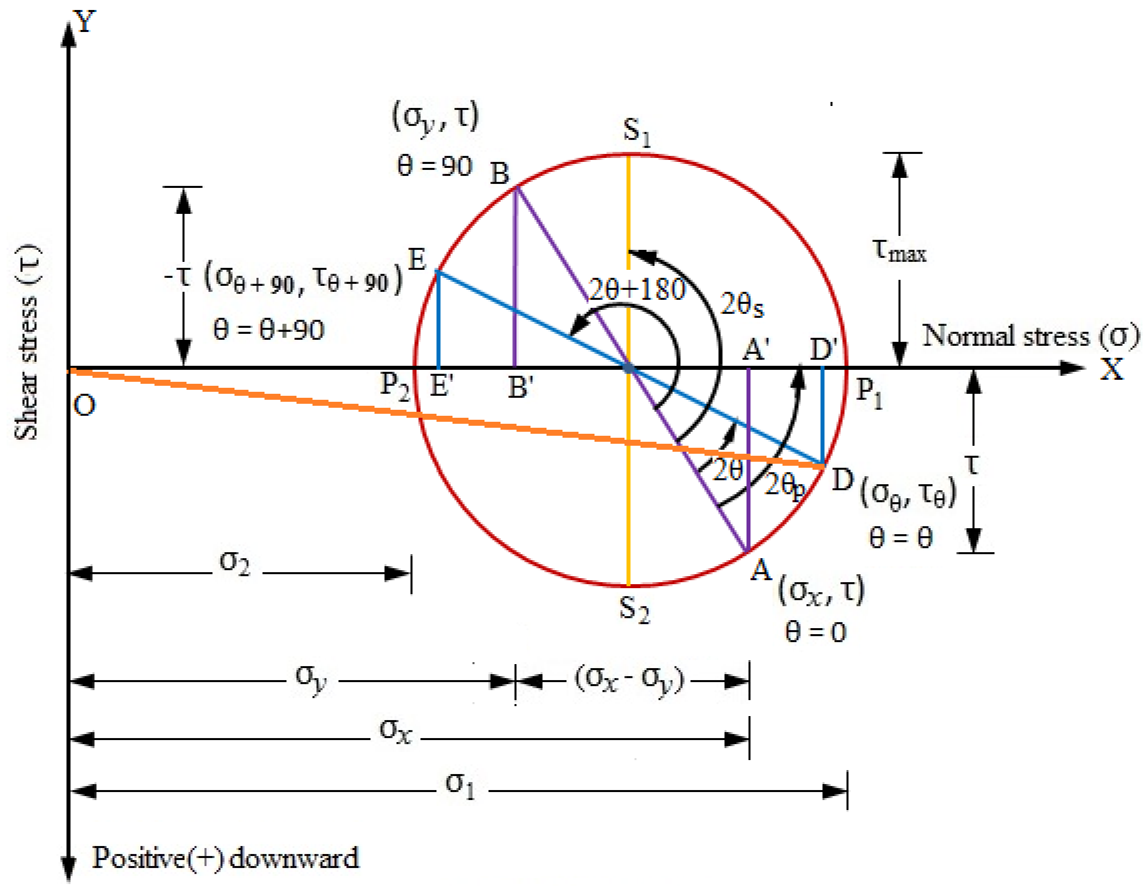


Fig. 15 Mohr's circle

Every point on the circumference of the circle then represents a state of stress on some plane through *C*.

The stress state on an inclined plane with an angle θ is represented at point *D* on the Mohr's circle, which is measured an angle 2θ counter- clockwise from point *A* to show the coordinate at *D*.

Consider any point *D* on the circumference of the circle, such that *CD* makes an angle 2θ with *CA*, and drop a perpendicular from *D* to meet the σ axis at *D'*.

Coordinates of *D*:

$$OD' = OC + CD' = \frac{1}{2}(\sigma_x + \sigma_y) + R\cos(2\theta_p - 2\theta)$$

$$= \frac{1}{2}\left(\sigma_x + \sigma_y\right) + R\cos 2\theta_p \cos 2\theta + R\sin 2\theta_p \sin 2\theta$$

But $$R\cos 2\theta_p = \frac{1}{2}\left(\sigma_x - \sigma_y\right) \; and \; R\sin 2\theta_p = \tau_{xy}$$

Therefore, $$OD' = \frac{1}{2}\left(\sigma_x + \sigma_y\right) + \frac{1}{2}\left(\sigma_x - \sigma_y\right)\cos 2\theta + \tau_{xy} \sin 2\theta$$

On inspection this is seen to be eqn. (12) for the normal stress σ_θ on the plane inclined at θ to the vertical plane *AB*.

Similarly, $$DD' = R\sin\left(2\theta_p - 2\theta\right)$$

$$DD' = R\sin 2\theta_p \cos 2\theta - R\cos 2\theta_p \sin 2\theta$$

$$DD' = -\frac{1}{2}\left(\sigma_x - \sigma_y\right)\sin 2\theta + \tau_{xy} \cos 2\theta$$

Again, on inspection this is seen to be eqn. (13) for the shear stress τ_θ on the plane inclined at θ to the vertical plane *AB*.

Thus the coordinates of ***Q*** are the normal and shear stresses on a plane inclined at θ to *AB* in the original stress system.

Characteristics of Mohr's circle

1. The direct stress is maximum when *D* is at P_1, i.e. OP_1 is the length representing the maximum principal stress σ_1 and $2\theta_p$ gives the angle of the plane θ_p, from *AB*. Similarly, OP_2 is the other principal stress.
2. The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle. This follows since shear stresses and complementary shear stresses have the same value; *therefore the centre of the circle will always lie on the* σ-*axis midway between* σ_x and σ_y.

3. From the above point the direct stress on the plane of maximum shear must be midway between σ_x and σ_y, i.e. $\frac{1}{2}(\sigma_x + \sigma_y)$.
4. The shear stress on the principal planes is zero.
5. Since the resultant of two stresses at 90^0 can be found from the parallelogram of vectors as the diagonal, as shown in Fig. **13.10,** the resultant stress on the plane at θ to AB is given by OD on Mohr's circle.
6. Comparing the Mohr's circle and the stress element, it is observed points S_1 and S_2 representing the points of maximum and minimum shear stresses, are located on the circle at 90^0 from points P_1 and P_2 i.e. the planes of maximum and minimum shear stress are at 45^0 to the principal planes, and

The graphical method of solution of complex stress problems using Mohr's circle is a very powerful technique since all the information relating to any plane within the stressed element is contained in the single construction. It thus provides a convenient and rapid means of solution which is less prone to arithmetical errors and is highly recommended.

Numerical

1. A material has permissible stresses in tension, compression and shear as 30 *N/mm*^{2}, 90 *N/mm*^{2} and 25 *N/mm*^{2} respectively. If specimens of diameter 20 *mm* are tested in tension and compression, identify the failure surfaces and failure load.

Solution:

Case I (Test in tension): If subjected to full tensile strength,

Maximum tensile stress, $\sigma_t = 30\, N/mm^2$

Corresponding maximum stress in shear, $\tau_{\max} = \dfrac{\sigma_t}{2} = \dfrac{30}{2} = 15\, N/mm^2 < 25\, N/mm^2$

Hence, the failure will occur due to tension. The maximum tensile stress is in the axial direction. Hence failure will occur on the plane of axial tensile stress, i.e., at right angle to the stress.

Corresponding tensile force, $P_t = \text{Area of specimen} \times \text{tensile strength}$

$$= A \times \sigma_t = \frac{\pi}{4} \times 20^2 \times 30$$

$$= 9224.778\, N$$

Case II (Test in compression):

Maximum compressive stress, $\sigma_c = 90\, N/mm^2$

Corresponding maximum stress in shear, $\tau_{\max} = \dfrac{\sigma_c}{2} = \dfrac{90}{2} = 45\, N/mm^2 > 25\, N/mm^2$

Hence, the failure will occur due to shear. The failure will occur on the plane at 45 to the plane of axial stress.

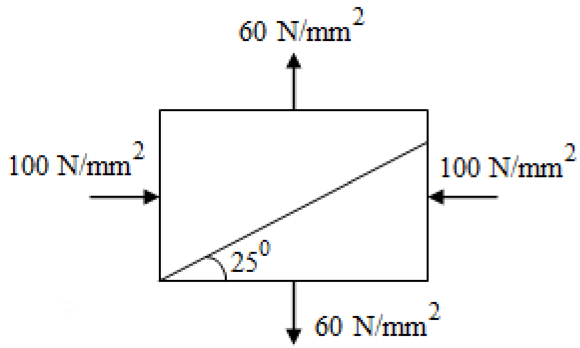
Corresponding compressive stress causing shear failure, $\sigma_{cf} = 2 \times 25 = 50\, N/mm^2$

Corresponding compressive force, $P_c = \text{Area of specimen} \times \text{compressive stress}$

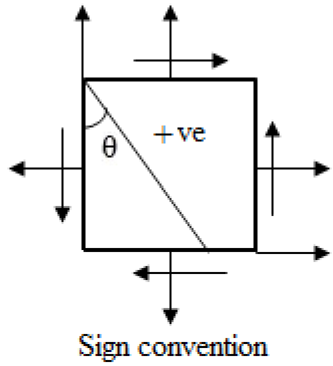
$$= A \times \sigma_{cf} = \frac{\pi}{4} \times 20^2 \times 50$$

$$= 15707.93\, N$$

2. Normal stresses acting at a point in a strained material are 100 *N/mm*^{2} compressive and 60 *N/mm*^{2} tensile as shown in the figure. Find the stresses on the given oblique plane.



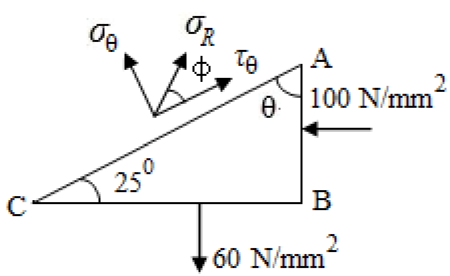
Solution: Sign convention:



Define the stresses in terms of the sign convention:

$\sigma_x = -100\,MPa,\ \sigma_y = 60\,MPa$

Angle of orientation of plane of plane AC, θ = - 65^0 (clockwise)



Normal stress on the oblique plane, $\sigma_\theta = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta$

$$= \frac{-100+60}{2} + \frac{-100-60}{2}\cos 2\left(-65^0\right)$$

$$= -20 - 80\cos\left(-130^0\right)$$

$$= 31.423\ N/mm^2\ \text{(tensile)}$$

Shear stress on the oblique plane, $\tau_\theta = -\dfrac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta$

$$= -\frac{(-100-60)}{2}\sin 2\left(-65^0\right)$$

$$= 80\sin\left(-130^0\right)$$

$$= -61.284\, N/mm^2 \text{ (Clockwise)}$$

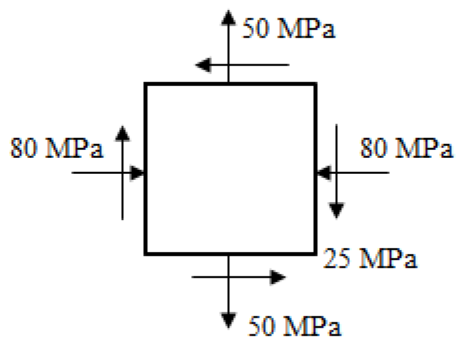
Resultant stress on the plane, $\sigma_R = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{31.432^2 + (-61.284)^2}$

$$= 68.87\, N/mm^2$$

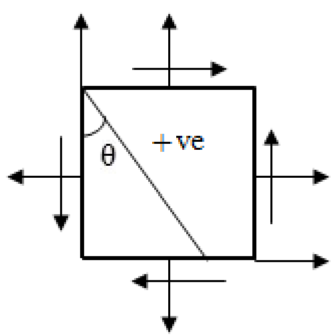
Angle of obliquity, $\phi = \tan^{-1}\left(\frac{\sigma_\theta}{\tau_\theta}\right) = \tan^{-1}\left(\frac{31.423}{61.284}\right)$

$$= 27.14^0$$

3. The state of plane stress at a point is represented by the stress element below. Determine the stresses acting on plane oriented 30° clockwise with respect to the vertical plane.



Solution: Sign convention:



Sign convention

Define the stresses in terms of the sign convention:

$$\sigma_x = -80\,MPa,\ \sigma_y = 50\,MPa \quad and \ \tau_{xy} = -25\,MPa$$

Angle of orientation of plane, θ = - 30^0

Normal stress on the given plane, $\sigma_\theta = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

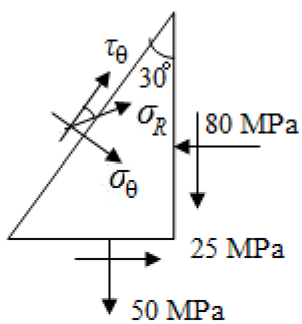
$$= \frac{-80+50}{2} + \frac{-80-50}{2}\cos 2(-30) + (-25)\sin 2(-30)$$

$$= \frac{-30}{2} + \frac{-130}{2}\cos(-60) - 25\sin(-60)$$

$$= -15 - 65\cos(-60) - 25\sin(-60)$$

$$= -15 - 65\cos(-60) - 25\sin(-60)$$

$$= -25.9\,MPa \ \ (\text{Compressio n})$$



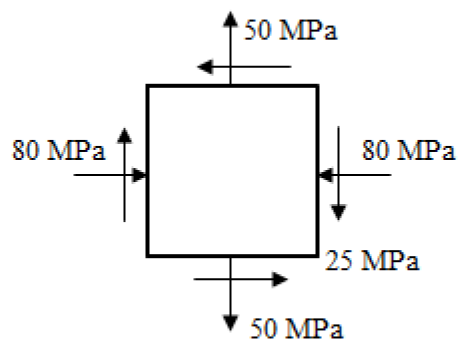
Shear stress on the oblique plane, $\tau_\theta = -\dfrac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$

$$= -\frac{(-80-50)}{2}\sin 2(-30^0) + (-25)\cos 2(-30^0)$$

$$= 65\times\sin(-60^0) - 25\cos(-60^0)$$

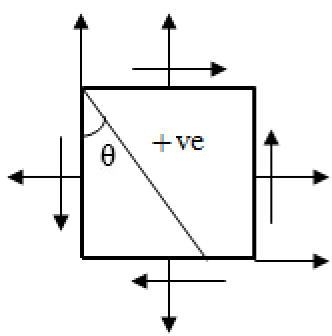
$$= -68.80\,MPa \quad (\text{Clockwise})$$

4. Example: The state of plane stress at a point is represented by the stress element below. Determine the principal stresses and draw the corresponding stress element.



Solution:

Sign convention:



Sign convention

Define the stresses in terms of the sign convention:

$$\sigma_x = -80\,MPa,\ \sigma_y = 50\,MPa \quad and\ \tau_{xy} = -25\,MPa$$

Angle of orientation of plane, θ = - 30^0

Principal stresses, $\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$

$$= \frac{1}{2}(-80+50) \pm \frac{1}{2}\sqrt{(-80-50)^2 + 4(-25)^2}$$

$$= -15 \pm \frac{1}{2}\sqrt{(-130)^2 + 4\times 625}$$

$$= -15 \pm 69.64$$

$$\sigma_1 = 54.64\, MPa \;\; and \;\; \sigma_2 = -84.64\, MPa$$

Angle of principal plane, $\tan 2\theta_p = \dfrac{2\tau_{xy}}{(\sigma_x - \sigma_y)} = \dfrac{2(-25)}{(-80-50)}$

$$\tan 2\theta_p = 0.3846$$

$$2\theta_p = 21.0^0 \text{ and } 21.0^0 + 180^0$$

$$\theta_p = 10.5^0 \text{ and } 100.5^0$$

Check for which angle goes with which principal stress. Put $\theta = 10.5^0$ in the stress equation,

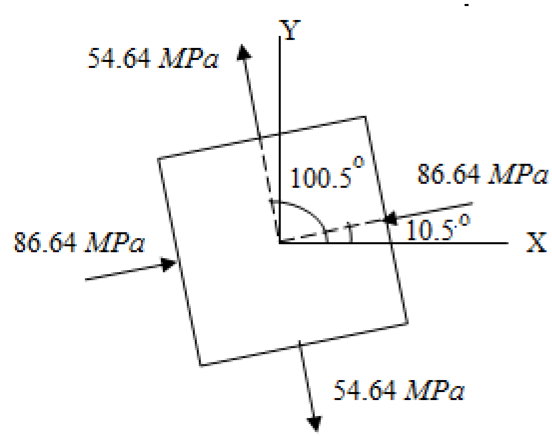
$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \frac{-80+50}{2} + \frac{-80-50}{2}\cos 2(10.5^0) + (-25)\sin 2(10.5^0)$$

$$= -84.6 \; MPa$$

$$\sigma_1 = 54.64\, MPa \; with\, \theta_{p1} = 100.5^0$$

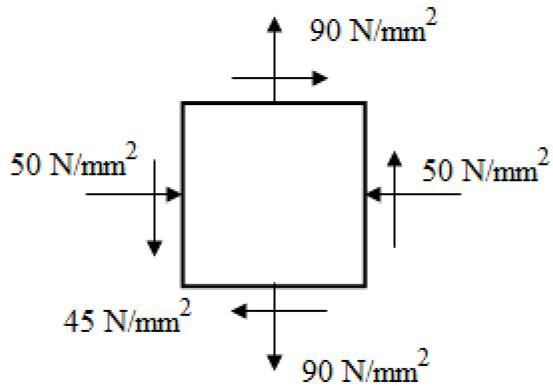
$$\sigma_2 = -84.64\, MPa \; with\, \theta_p = 10.5^0$$



5. A point in a stressed structural member is subjected to a tensile stress of 90 N/mm^2 on a horizontal plane and compressive stress of 50 N/mm^2 on the vertical plane. There is a shear

stress of 45 *N/mm*^{2} such that when on vertical plane, it tends to rotate the member in counterclockwise direction. Determine the maximum shear stress and also the resultant stress on the planes of maximum shear stresses.

Solution:



Define the stresses in terms of the sign convention:

$$\sigma_x = -50\,MPa,\ \sigma_y = 90\,MPa \quad and\ \tau_{xy} = 45\,MPa$$

Angle of orientation of plane, θ = - 30^0

Maximum shear stress, $\tau_{\max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$

$$= \frac{1}{2}\sqrt{(-50-90)^2 + 45^2} = 73.53\,N/mm^2\ (counterclockwise)$$

Inclination of plane of maximum shear to the vertical plane,

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}} = -\frac{(-50-90)}{2\times 45}$$

$$\tan 2\theta_s = -1.555$$

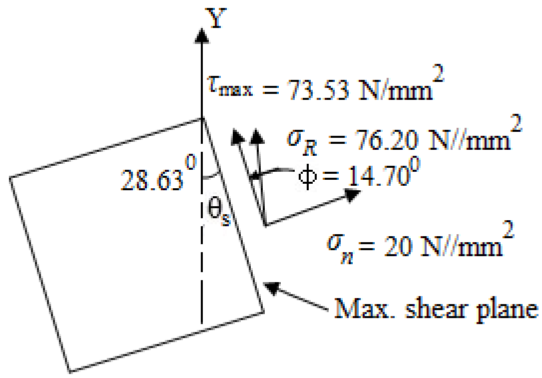
$$2\theta_s = \tan^{-1}(1.555) = 57.26^0 \quad and\ 237.26^0$$

$$\theta_s = 28.63^0\ and\ 118.63^0 \qquad (counterclockwise)$$

Normal stress on plane of maximum shear stress,

$$\sigma_n = \frac{-50+90}{2} + \frac{-50-90}{2}\cos 2\left(28.63^0\right) + 45\sin 2\left(28.63^0\right)$$

$$= 20\,N/mm^2$$



Resultant stress on the plane, $\sigma_R = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{20^2 + 73.53^2}$

$$= 76.20\,N/mm^2$$

$$\phi = \tan^{-1}\left(\frac{\sigma_n}{\tau_{\max}}\right) = \tan^{-1}\left(\frac{20}{76.20}\right)$$

$$= 14.70^0$$

Summary

1. The normal stress **σ** and shear stress τ on oblique planes resulting from direct loading are

$$\sigma_\theta = \sigma\cos^2\theta$$

$$\tau_\theta = \frac{\sigma}{2}\sin 2\theta$$

2. The stresses on oblique planes owing to a complex stress system are:

Normal stress, $\sigma_\theta = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

Normal stress, $\tau_\theta = -\dfrac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$

3. The ***principal stresses*** (i.e. the maximum and minimum direct stresses) are then

$$\sigma_{1,2} = \frac{1}{2}\left(\sigma_x + \sigma_y\right) \pm \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2}$$

and these occur on planes at an angle θ_p to the vertical plane on which σ_x, acts, given by either

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\left(\sigma_x - \sigma_y\right)}$$

where $\sigma_p = \sigma_1$ or σ_1**,** the planes being termed ***principal planes.*** The principal planes are always at 90^0 to each other, and the ***planes*** of ***maximum shear*** are then located at 45^0 to them.

4. The ***maximum shear stress*** is

$$\tau_{\max} = \pm\frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2}$$

$$\tau_{\max} = \pm\frac{1}{2}\left(\sigma_1 - \sigma_2\right)$$

5. Normal stress on plane of maximum shear $= \frac{1}{2}\left(\sigma_x + \sigma_y\right)$
6. Shear stress on plane of maximum direct stress (principal plane) = 0
7. In problems where the principal stress in the third dimension σ_3 either is known or can be assumed to be zero, the true maximum shear stress is then

$$\frac{1}{2}\left(greatest principal\ stress - least\ principal\ stress\right)$$

Mohr's circle:

Mohr's circle is a graphical representation of stress transformation equations. The equations of stress transformation describe a circle if normal stress and shear stress are represented as abscissa and ordinate respectively. Each point on the circumference of Mohr's circle represents a plane through the centre of the circle and the coordinates (σ, τ) of the point represents the normal stress (σ) and shear stress (τ) on the given plane.

Mohr's circle can be drawn from a given state of stress at a point in a structural member. Consider a stress element representing the state of stress at a point as shown in the figure.

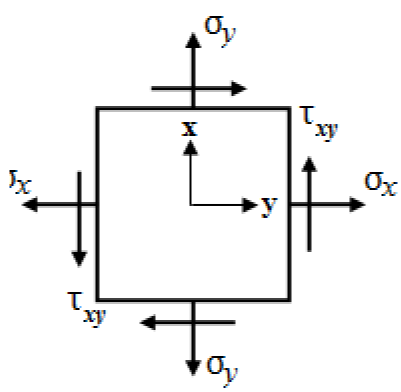


Fig.

Stress transformation equations for normal and tangential components on a plane are given by

Normal stress on the plane, $$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (12)$$

Shear stress on the plane, $$\tau_\theta = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (13)$$

Rearranging the equation (12), we have

$$\sigma_\theta - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (14)$$

Squaring both sides of equation (13) and (14) and adding them together, we have

$$\left(\sigma_\theta - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\theta^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$\left(\sigma_\theta - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\theta^2 = \left(\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\right)^2$$

This is the equation of a circle with centre $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$ and radius $R = \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ and this circle is known as Mohr's circle named after the German Civil Engineer Otto Mohr (1835-1918). It provides a simple and clear picture of an otherwise complicated analysis.

Procedure for drawing Mohr's circle:

1. Draw coordinates axes in Cartesian coordinate system with O as origin, normal stress (σ) as abscissa (positive to the right) and shear stress (τ) as ordinate (positive upward).
2. Locate the centre C of the circle at the point having coordinates $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$.
3. Locate point *A*, representing the state of stress on the vertical plane, i.e., face *x* of the element by plotting its coordinates σ_x and τ. Point *A* on the circle corresponds to $\theta = 0^0$ and represents the vertical plane.
4. Locate point *B*, representing the state of stress on the horizontal plane, i.e., face y of the element by plotting its coordinates σ_y and $-\tau$. Point *A* on the circle corresponds to $\theta = 90^0$ and represents the horizontal plane.
5. Join AB so as to intersect the normal stress axis at *C*.
6. With the point *C* as the centre and *CA* (= *CB*) as radius, draw Mohr's circle through points *A* and *B*. This is the required Mohr's circle which has radius *R*.

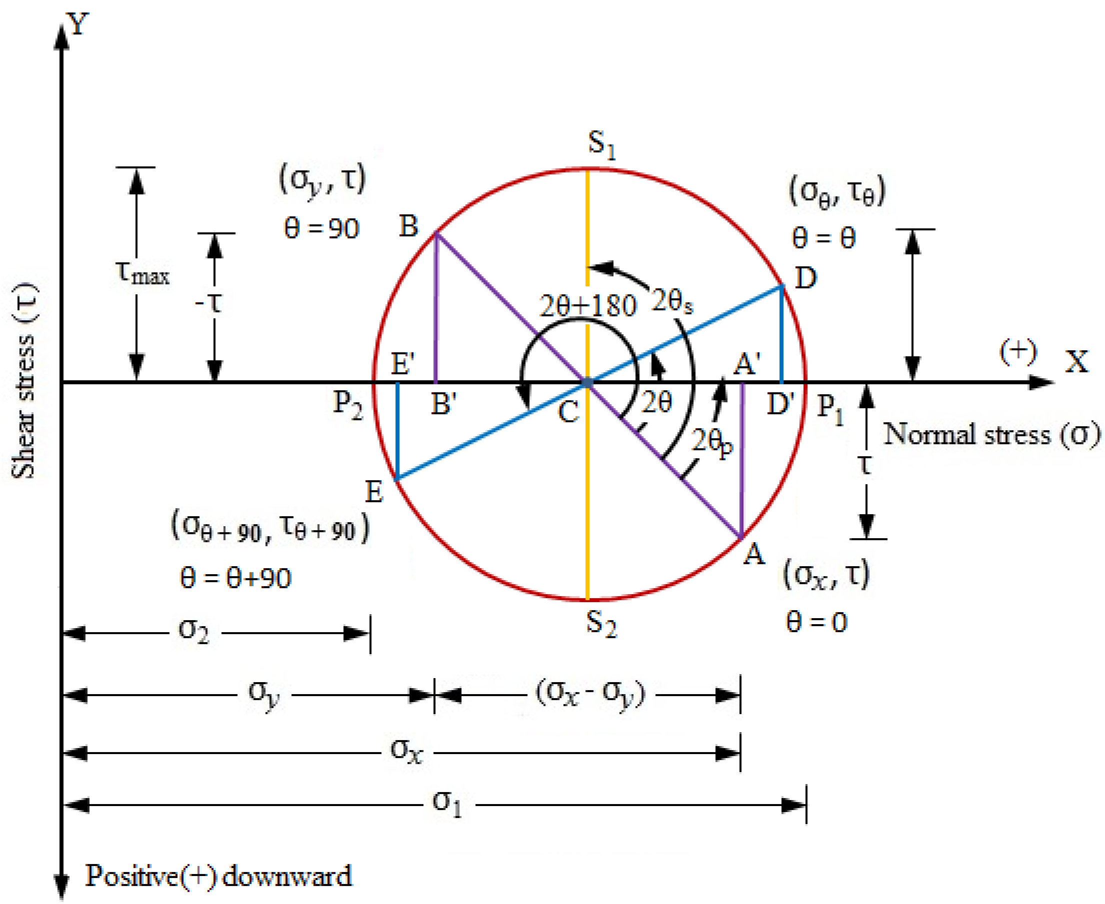


Fig. Mohr's circle

The stress state on an inclined plane with an angle θ is represented at point D on the Mohr's circle, which is measured an angle 2θ counter- clockwise from point A to show the coordinate at D.

Coordinates of D:

$$OD' = OC + CD' = \frac{1}{2}\left(\sigma_x + \sigma_y\right) + R\cos\left(2\theta - 2\theta_p\right)$$

$$= \frac{1}{2}\left(\sigma_x + \sigma_y\right) + R\cos 2\theta \cos 2\theta_p + R\sin 2\theta \sin 2\theta_p$$

But $$R\cos 2\theta_p = \frac{1}{2}\left(\sigma_x - \sigma_y\right) \text{ and } R\sin 2\theta_p = \tau_{xy}$$

Therefore, $$OD' = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy} \sin 2\theta$$

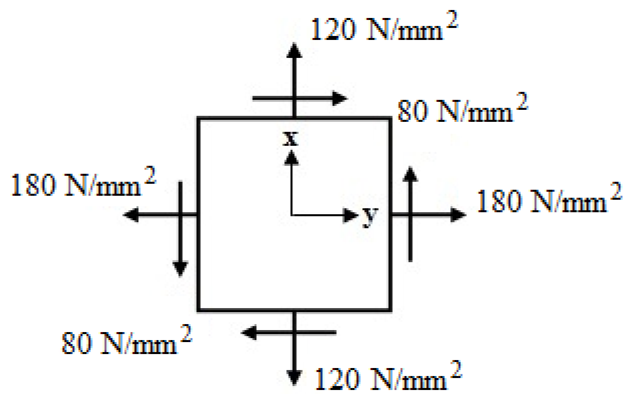
Similarly, $$DD' = R\sin(2\theta - 2\theta_p)$$

$$DD' = R\sin 2\theta \cos 2\theta_p - R\cos 2\theta \sin 2\theta_p$$

$$DD' = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy} \cos 2\theta$$

Problem. The state of stress in a material is as shown in the figure. Determine

a. the magnitude and directions of principal stresses, and
b. the magnitude of maximum shear stresses and its direction.



Numerical

1. A rectangular beam of breadth 100 *mm* and depth 200 *mm* is simply supported over a span of 4 *m*. The beam is loaded with an uniformly distributed load of 5 *kN/m* over the entire span. Find the maximum bending stresses.

Solution:

Breadth of the beam, b = 100 *mm*

Depth of beam, d = 200 *mm*

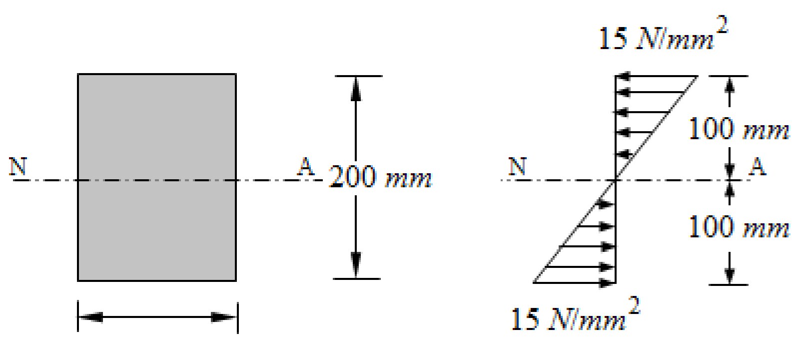
Moment of inertia, $I = \frac{1}{12}bd^3 = \frac{1}{12}\times 100\times (200)^3 = 66.67\times 10^6\ mm^4$

Span of beam, l = 4 *m*

Uniformly distributed load, w = 5 *kN/m*

Maximum bending moment at centre of beam, $M = \frac{wl^2}{8} = \frac{5\times 4^2}{8}$

$$= 10\,kN.M = 10^7\ N.mm$$



(a) Cross section of beam (b) Bending stress distribution

Fig. 22

Neutral axis passes through the centroid of section.

The distance of top and bottom fibre from the neutral axis, y = 100 *mm*

Thus, maximum bending stress, $\sigma = \frac{M}{I}y = \frac{10^7}{66.67\times 10^6}\times 100$

$$= 15\, N/mm^2$$

2. A beam of I-section shown in Fig. 23 is simply supported over a span of 10 *m*. It carries a uniform load of 4 *kN*/*m* over the entire span. Evaluate the maximum bending stresses.

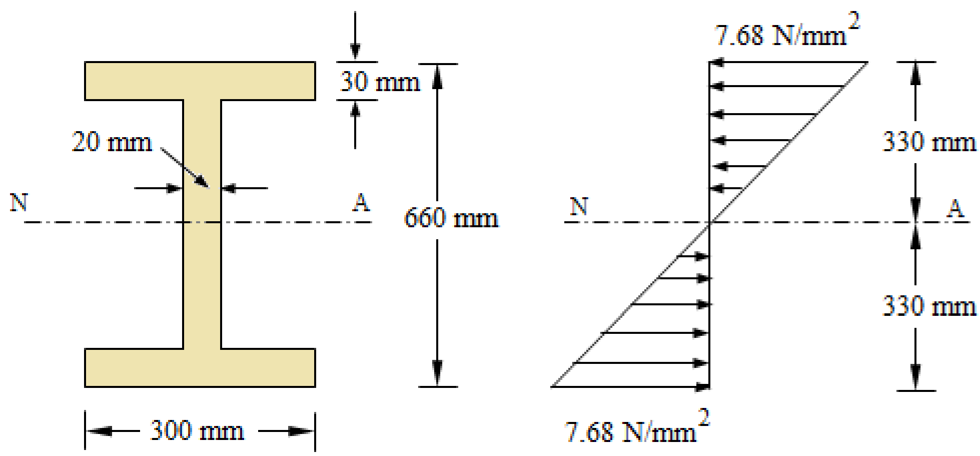
Solution:

Moment of inertia, $I = \frac{1}{12}\left(BD^3 - bd^3\right) = \frac{1}{12}\left(300 \times 660^3 - 280 \times 600^3\right)$

$$= 21.474 \times 10^8\ mm^4$$

Span of the beam, $l = 10\ m$

Uniformly distributed load, $w = 4\ kN/m$



(a) Cross section of beam (b) Bending stress distribution

Fig. 23

Maximum bending moment at centre of beam, $M = \frac{4 \times 10^2}{8} = 50\, kN.m$

$$= 5 \times 10^7\ N.mm$$

Neutral axis passes through the centroid of I-section.

The distance of top and bottom fibre from the neutral axis, $y = 330\ mm$

Thus, maximum bending stress, $\sigma = \frac{M}{I} y = \frac{5\times10^{7}}{21.474\times10^{8}} \times 330 = 7.68\, N/mm^{2}$

The bending stress at top and bottom fibres $= 7.68\times10^{8}\; N/mm^{2}$

3. A beam of an I-section shown in Fig. 24 is simply supported over a span of 4 *m*. Find the uniformly distributed load the beam can carry if the bending stress is not to exceed 100 N/mm^2.

Solution:

Moment of inertia, $I = \frac{1}{12}\left(BD^{3} - bd^{3}\right) = \frac{1}{12}\left(200\times300^{3} - 180\times260^{3}\right)$

$$= 180.36\times10^{6}\; mm^{4}$$

Maximum bending stress, $\sigma_{\max}$ = 100 *N/mrn*^{2}

Span of beam, l = 4 *m*

Extreme fibre distance, $y_{\max}$ = 150 *mm*

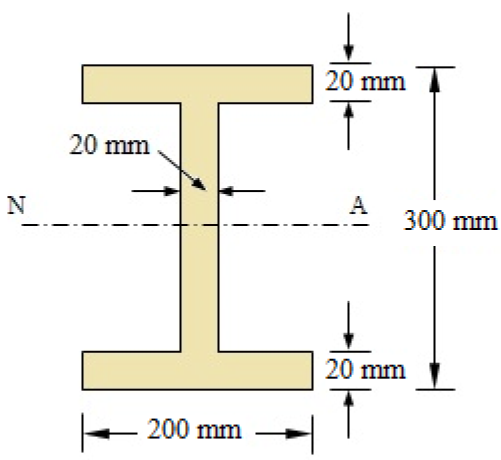


Fig. 24

Section modulus, $Z = \frac{I}{y_{\max}} = \frac{180.36\times10^{6}}{150} = 1242400\, mm^{3}$

Maximum bending moment, $M = \sigma_{\max} Z = 100\times1242400$

$$= 124240000\, N.mm$$

$$= 124.24\, kN.m$$

But $$M = \frac{wl^2}{8}$$

$$124.24 = \frac{w \times (4)^2}{8}$$

$$w = \frac{124.24 \times 8}{16} = 64.12\, kN/m$$

The maximum uniformly distributed load the beam can carry = 64.12 *kN*/*m*.

4. A timber beam of rectangular section carries a load of 2 *kN* at mid-span. The beam is simply supported over a span of 3.6 *m*. If the depth of section is to be twice the breadth, and the bending stress is not to exceed 9 *N*/*mrn*^{2}, determine the cross-sectional dimensions.

Solution:

Span of the beam, l = 3.6 *m*

Uniformly distributed load, w = 2 *kN*

Allowable bending stress, σ_{allow} = 9 *N*/*mm*^{2}

Maximum bending moment at centre of beam, $M = \frac{WL}{4} = \frac{2 \times 3.6}{4} = 1.8\, kN.m$

$$= 1.8 \times 10^6\ N.mm$$

From the flexural relationship, we have $$Z = \frac{M}{\sigma_{allow}}$$

$$\frac{1}{6} bd^2 = \frac{1.8 \times 10^6}{9}$$

$$bd^2 = \frac{1.8 \times 10^6}{9} \times 6 = 1.2 \times 10^6$$

Depth of section is to be twice the breadth, i.e., $d = 2b$

So, we have $$b(2b)^2 = 1.2 \times 10^6$$

$$b^3 = \frac{1.2\times10^6}{4} = 0.3\times10^6$$

$$b = 64.94\,mm$$

$$d = 2\times64.943 = 129.886\,mm$$

Therefore, width of beam = 65 *mm*, and depth of beam = 130 *mm*

5. A rectangular beam of width 200 *mm* and depth 300 *mm* is simply supported over a span of 5 *m*. Find the safe uniformly distributed load that the beam can carry per metre length if the allowable bending stress in the beam is 100 N/mm^2.

Solution:

Span of beam, l = 5 *m*

Width Breadth of the beam, b = 100 *mm*

Depth of beam, d = 200 *mm*

Allowable bending stress, σ_{allow} = 100 N/mm^2

Section modulus, $Z = \frac{1}{6}bd^2 = \frac{1}{2}\times200\times300^2 = 3\times10^6\ mm^3$

Moment of resistance of the beam, $M = \sigma_{allow}Z = 100\times3\times10^6$

$$= 300\times10^6\ N.mm = 300\,kN.m$$

Maximum bending moment at the centre of the beam,

$$M = \frac{wl^2}{8}$$

$$300 = \frac{w\times(5)^2}{8}$$

$$\therefore \qquad w = \frac{300\times8}{25} = 96\,kN.m$$

So, the load that the beam can carry is 96 kN/m.

6. A rectangular beam of size 60 *mm* x 100 *mm* has a central rectangular hole of size 15 *mm* x 20 *mm*. The beam is subjected to bending and the maximum bending stress is limited to 100 N/mm^2. Find the moment of resistance of the hollow beam section.

Solution:

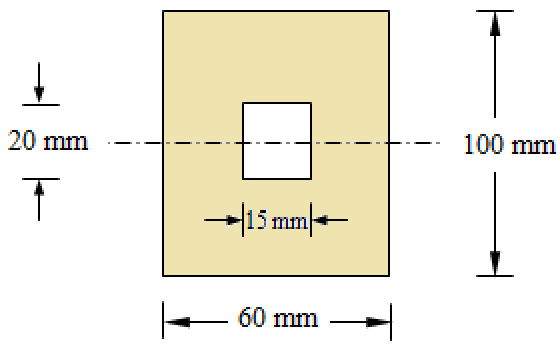


Fig.25