DEPARTMENT OF MATHEMATICS & SC

ENGINEERING MATHEMATICS-III

DIPLOMA 3RD SEMESTER

LEARNING MATERIAL

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Unit-2 Complex Number

$* \quad x^2+1=0$

$x^2=-1$

$x=\sqrt{-1}$

$i^2=i.i=\sqrt{-1}.\sqrt{-1}=-1$

$i^3=i^2.i=-1\times i=-i$

$i^4=i^2.i^2=-1\times-1=1$

$i^4=1=i^8=i^{12}=i^{16}\ldots\ldots=i^{4n}$

$i^3=-i=i^7=i^{11}=i^{15}\ldots\ldots=i^{4n-1}$

$i^2=-1=i^6=i^{10}=i^{14}\ldots\ldots=i^{4n-2}$

$i=\sqrt{-1}=i^5=i^9=i^{13}\ldots\ldots i^{4n-3}$

Defination

$\Rightarrow$ The number which is expressed in the form of $x+iy$ / $a+ib$, where $a,b\in R$ or $x,y\in R$ and $i=\sqrt{-1}$

e.g:- $6+0i$, $5-3i$, $2i$

$\Rightarrow$ The complex number is denoted by "z". i.e. $z=a+ib$

$\Rightarrow$ Each complex number have two parts, One is Real (R) and other is Imaginary (I).

If $z=x+iy$,

Then, $x\rightarrow$ Real part

$y\rightarrow$ Imaginary part

$R(z)=x$ or $I(z)=y$.

$\Rightarrow$ If a complex number is purely Real then it's imaginary part is zero.

e.g - $z=2,3,-4\ldots\ldots$ etc

$\Rightarrow$ If a complex number is purely Real then it's real part is zero.

eg. $-2i, 3i\ldots\ldots$ etc..

Algebraic Sum of two complex no.

Let $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$\therefore z_1 + z_2 = \underbrace{(x_1+x_2)}_{\text{Real}} + \underbrace{i(y_1+y_2)}_{\text{Imaginary}}$

$z_1 = 3+5i$

$z_2 = 2-3i$

$z_1 + z_2 = (3+2)+(5-3)i$

$= 5+2i$

Difference of two complex no.

Let, $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

$z_1 = 3-5i$

$z_2 = 1+2i$

$z_1 - z_2 = (3-1)+i(-5-2)i$

$= 2+(-7)i$

$= 2-7i$

* Multiplication of two complex No.

Let, $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$

than $z_1 z_2 = (a_1 + ib_1)$

$= a_1(a_2 + ib_2) + ib_1(a_2 + ib_2)$

$= a_1 a_2 + ia_1 b_2 + ib_1 a_2 + i^2 b_1 b_2$

$= a_1 a_2 + ia_1 b_2 + ib_1 a_2 - b_1 b_2$

$= \underbrace{(a_1 a_2 - b_1 b_2)}_{R} + \underbrace{i(a_1 b_2 + b_1 a_2)}_{I}$

∴ Multiplication of 2 complex No. is a complex No.

* Division of two complex No

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$\because \dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} \times \dfrac{x_2 - iy_2}{x_2 - iy_2}$

Real no.

$= \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2)^2 - (iy_2)^2}$

$= \dfrac{x_1(x_2 - iy_2) + iy_1(x_2 - iy_2)}{x_2^2 - i^2 y_2^2}$

$= \dfrac{x_1 x_2 - ix_1 y_2 + iy_1 x_2 - i^2 y_1 y_2}{x_2^2 - i^2 y_2^2}$

$= \dfrac{x_1 x_2 - ix_1 y_2 + ix_2 y_1 + y_1 y_2}{x^2 + y_2^2}$

$= \dfrac{x_1 x_2 + y_1 y_2 + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} + i\dfrac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$

Conjugate of complex No.

⇒ The conjugate of a complex No. is obtained from a complex No by changing the sign of imaginary parts.
⇒ It is denoted by the symbol '$\bar{z}$'.
⇒ Double conjugate of a complex number remain gives same complex Number.

e.g :- $z = 2-5i$
Ex-1 $\bar{z} = 2+5i$ complex conjugate
$\overline{(\bar{z})} = 2-5i$

Ex-2 $z_2 = -3+4i$
$\bar{z}_2 = -3-4i$

properties of current

① $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

e.g. $z_1 = 4+3i$
$z_2 = 6+4i$

$\overline{z_1+z_2}$

L.H.S
$z_1+z_2 = 4+3i+6+4i$
$= 10+7i$
$\overline{z_1+z_2} = 10-7i$

R.H.S
$\bar{z}_1 = 4-3i$
$\bar{z}_2 = 6-4i$
$\bar{z}_1+\bar{z}_2 = 4-3i+6-4i$
$= 10-7i$

L.H.S = R.H.S

(2) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(3) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Imp (4) $z \cdot \bar{z} = |z|^2$

(5) $R(z) = \frac{1}{2}(z+\bar{z})$

$z = 5-4i$
$\bar{z} = 5+4i$

$z+\bar{z} = 10$
$\therefore \frac{1}{2}(z+\bar{z}) = \frac{1}{2} \times 10 = 5$
L.H.S = R.H.S

(6) $Im(z) = \frac{1}{2i}(z-\bar{z})$

* Modulus of a complex No.-

If $z = a+ib$ be a complex No., then the modulus of z written as $|z|$ is a real number.

e.g:- $z = a+ib$

$|z| = \sqrt{a^2+b^2}$

(i) $z = 3+2i$

$|z| = \sqrt{(3)^2+(2)^2}$

Properties

(i) $|z| = |\bar{z}|$

e.g:- $z = 3+4i$

$|z| = \sqrt{(3)^2+(4)^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5 \in R$

$\bar{z} = 3-4i$

$|\bar{z}| = \sqrt{(3)^2+(-4)^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5 \in R$

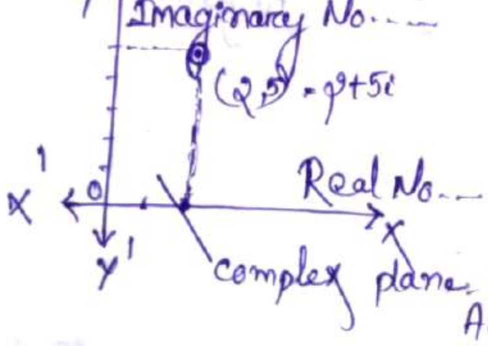
Complex Number as an order pair :-

→ We know the complex number is $x+iy$, where $x, y \in R$. The complex number uniquely represented by a order pair of (x,y).

e.g.. $z = (3,4) = 3+4i$

$z = (2,-3) = 2-3i$

* Geometrical representation of a complex number

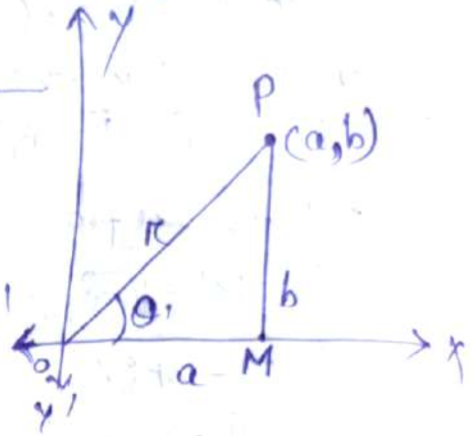


$z_1 = 2+5i$
$= (2,5)$

⇒ The plane on which a complex number are represented is known as complex plane or argand plane.

IMP

* Polar form of a complex number



$$z = a + ib$$
$$\cdot(a, b)$$

Here, $\sin\theta = P/h = b/r$.

$$\cos\theta = b/h = a/r$$

from equation (i):-

$$a = r\cos\theta$$
$$b = r\sin\theta$$

Squaring both equation and adding.

$$a^2 = r^2\cos^2\theta$$
$$b^2 = r^2\sin^2\theta$$

$$a^2+b^2 = r^2(\sin^2\theta+\cos^2\theta)$$
$$a^2+b^2 = r^2(1)$$
$$a^2+b^2 = r^2$$
$$\boxed{r = \sqrt{a^2+b^2} = |z|}$$

Diving both equation (1) with (2)

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{b}{r}}{\frac{a}{r}} = \frac{b}{a}$$

$$\Rightarrow \tan\theta = \frac{b}{a}.$$

$$\boxed{\Rightarrow \theta = \tan^{-1} b/a}$$ Amplitude or Argument

$z = a + ib$

polar form

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

Ex-1

Convert the complex number (1+i) into polar form.

Soln

let, $z = 1+i = r(\cos\theta + i\sin\theta)$ ——— (1)

$r = \sqrt{(1)^2+(1)^2}$

$= \sqrt{1+1}$

$= \sqrt{2}$

$= \sqrt{2}$.

$\theta = \tan^{-1}\frac{1}{1} = \tan^{-1}1$

$\theta = 45 = \pi/4$.

∴ Hence, $z = (1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(a) let, $z = -1+i\sqrt{3} = r(\cos\theta + i\sin\theta)$ ——— (1)

$r = \sqrt{(-1)^2+(\sqrt{3})^2}$

$= \sqrt{1+3}$

$= \sqrt{4} = 2$.

$\theta = \tan^{-1}\frac{\sqrt{3}}{-1}$

$\theta = \tan^{-1}\frac{-\sqrt{3}}{1}$

$\theta = 120^\circ = \frac{2\pi}{3}$

∴ Hence, $z = (-1+i\sqrt{3}) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Inverse of a complex No. –

$z = a+ib$ be a complex Number.
Then their inverse is denoted by "z^{-1} or $1/z$".

Soln

$z = a+ib$

$$\frac{1}{z} = \frac{1}{a+ib}$$

$$= \frac{1\,(a-ib)}{(a+ib)(a-ib)}$$

$$= \frac{a-ib}{(a)^2 - (ib)^2} = \frac{a-ib}{a^2 - i^2b^2} = \frac{a-ib}{a^2-(-1)b^2} = \frac{a-ib}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} - i\left(\frac{b}{a^2+b^2}\right)$$

e.g. – Find the inverse of complex No.

$z = 1+i$

Soln

$z = 1+i$

$$\frac{1}{z} = \frac{1}{1+i}$$

$$= \frac{1\,(1-i)}{(1+i)(1-i)}$$

$$= \frac{1-i}{(1)^2-(i)^2} = \frac{1-i}{(1)^2-(-1)} = \frac{1-i}{1+1} = \frac{1-i}{2} = \frac{1}{2} - i\left(\frac{1}{2}\right)$$

e.g. – Find the value of $\frac{1}{(2+3i)^2}$

$$z = \frac{1}{(2+3i)^2} = \frac{1}{(2)^2+(3i)^2+2\cdot 2\cdot 3i}$$

$$= \frac{1}{4+9i^2+12i}$$

$$= \frac{1}{4+9\cdot(-1)+12i} = \frac{1}{4-9+12i} = \frac{1}{-5+12i}$$

$$= \frac{1\,(-5-12i)}{(-5+12i)(-5-12i)} = \frac{-5-12i}{(-5)^2-(12i)^2}$$

$$= \frac{-5-12i}{25-144i^2}$$

$$= \frac{-5-12i}{25+144}$$

$= \frac{-5-12i}{169}$

$= \frac{-5}{169} - i\left(\frac{12}{169}\right)$

e.g.-3 find the real and imaginary part of a complex no.

$Z = \left(\frac{3-2i}{4+3i}\right)$

$= \frac{3-2i\,(4-3i)}{(4+3i)(4-3i)}$

$= \frac{3(4-3i)-2i\,(4-3i)}{(4)^2-(3i)^2}$

$= \frac{12-9i-8i+6i^2}{16-9i^2}$

$= \frac{12-17i+6\times-1}{16-9\times-1}$

$= \frac{12-17i-6}{16-(-9)}$

$= \frac{\cancel{12-17i}\ 6-17i}{16+9}$

$= \frac{6-17i}{25}$

$= \frac{6}{25} - i\left(\frac{17}{25}\right)$

$R(z) = \frac{6}{25}$

$I(z) = \frac{-17}{25}$

Cube Root of a complex No. / cube root of unity

let

$\sqrt[3]{1} = x$

cubing both sides:-

$1 = x^3$

$x^3 - 1 = 0$ $\quad [a^3 - b^3 = (a-b)(a^2+ab+b^2)]$

$\Rightarrow (x-1)(x^2 + x \cdot 1 + 1^2) = 0$

$\Rightarrow (x-1)(x^2 + x + 1) = 0$

Either, $x - 1 = 0$ $\quad$ or $x^2 + x + 1 = 0$ - Quadratic Eqn.

$x = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a = 1, b = 1, c = 1$$

$$= \frac{-1 \pm \sqrt{(1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$\boxed{x = \frac{-1 + \sqrt{3}i}{2} = \omega} \quad \boxed{x = \frac{-1 - \sqrt{3}i}{2} = \omega^2}$$

$$\boxed{x = 1}$$

$\therefore$ Hence, $1, \omega, \omega^2$ are the cube root of unity.

Note-1 The sum of cube Root of unity is zero.

i.e.- $1+\omega+\omega^2=0$

(2) ~~The value of $\omega^3=1$~~

(3) from the Relation $1+\omega+\omega^2=0$

we get, $1+\omega=-\omega^2$

$1+\omega^2=-\omega$

$\omega+\omega^2=-1$

IMP

Express in the form of $a+ib$.

(i) $\dfrac{3+5i}{2-3i}$ (ii) $\dfrac{(1+i)^2}{3-i}$

$z=\dfrac{3+5i(2+3i)}{(2-3i)(2+3i)}$

$=\dfrac{3(2+3i)+5i(2+3i)}{(2)^2-(3i)^2}$

$=\dfrac{6+9i+10i+15i^2}{4-9i^2}$

$=\dfrac{6+19i+15i^2}{4-9\times-1}$

$=\dfrac{6+19i+15\times-1}{4+9}$

$=\dfrac{6+19i-15}{13}$

$=\dfrac{-9+19i}{13}$

$=-9/13+i\,\dfrac{19}{13}$

$=\dfrac{(1)^2+2.1.i+(i)^2}{3-i}$

$=\dfrac{1+2i+(-1)}{3-i}$

$=\dfrac{\cancel{1}+2i-\cancel{1}}{3-i}$

$=\dfrac{2i(3+i)}{(3-i)(3+i)}$

$=\dfrac{6i+2i^2}{(3)^2-(i)^2}$

$=\dfrac{6i+2(-1)}{9-(-1)}$

$=\dfrac{6i-2}{9+1}$

$=\dfrac{6i-2}{10}$

$=\left(\dfrac{6}{10}\right)i-\dfrac{2}{10}$

$=\dfrac{-2}{10}+i\left(\dfrac{6}{10}\right)$

$=-1/5+i\,3/5$

IMP

Express with rational denominator:-

(i) $\dfrac{1}{3-\sqrt{-5}}$

$= \dfrac{1}{3-\sqrt{5}i}$

$= \dfrac{1\,(3+\sqrt{5}i)}{(3-\sqrt{5}i)(3+\sqrt{5}i)}$

$= \dfrac{3+\sqrt{5}i}{(3)^2-(\sqrt{5}i)^2}$

$= \dfrac{3+\sqrt{5}i}{9-5i^2}$

$= \dfrac{3+\sqrt{5}i}{9-5\times -1}$

$= \dfrac{3+\sqrt{5}i}{9+5}$

$= \dfrac{3+\sqrt{5}i}{14}$

* Prove that

(i) $(1-\omega+\omega^2)^5+(1-\omega^2+\omega)^5=32$

(ii) $(2+5\omega+2\omega^2)^6=(2+2\omega+5\omega^2)^6=729$

(iii) $(1-\omega+\omega^2)^7+(1+\omega-\omega^2)^7=128$

L.H.S

* (i) $(1-\omega+\omega^2)^5+(1-\omega^2+\omega)^5$

$=(1+\omega^2-\omega)^5+(1+\omega-\omega^2)^5$

$=(-\omega-\omega)^5+(-\omega^2-\omega^2)^5$

$= \cancel{(-2)^5 +}$

$=(-2\omega)^5+(-2\omega^2)^5$

$= -32\omega^5 + -32\omega^{10}$

$= -32\,(\omega^5+\omega^{10})$

$= -32\,[\omega^3\cdot\omega^2+(\omega^3)^3\cdot\omega]$

$= -32\,[\omega^2+\omega]$

$= -32\,(-1)$

$= 32$

IMP

(ii) $(1-\omega+\omega^2)^7+(1+\omega-\omega^2)^7$

$=(1+\omega^2-\omega)^7+(1+\omega-\omega^2)^7$

$=(-\omega-\omega)^7+(-\omega^2-\omega^2)^7$

$=(-2\omega)^7+(-2\omega^2)^7$

$= -128\,\omega^7 + -128\,\omega^{14}$

$= -128\,(\omega^7+\omega^{14})$

$= -128\,[(\omega^3)^2\omega+(\omega^3)^4\cdot\omega^2]$

$= -128\,[1\cdot\omega+1\cdot\omega^2]$

$= -128\,[\omega+\omega^2]$

$= -128\,(-1) = 128$

$\dfrac{64}{\dfrac{2}{128}}$

IMP

(iii) $(1-\omega+\omega^2)+(1+\omega-\omega^2)=2$

$(1-\omega+\omega^2)+(1+\omega-\omega^2)$

$=(1+\omega^2-\omega)+(1+\omega-\omega^2)$

$=(-\omega-\omega)+(-\omega^2-\omega^2)$

$=(-2\omega)+(-2\omega^2)$

$\cancel{= -2\omega + -2}$

$\cancel{= -2(\omega+\omega^2)}$ $\cancel{=4\omega^3}$

$\cancel{= -2(\omega^3)}$ $\cancel{=4\times 1}$

$\cancel{= -2\times 1}$ $=$

$= 2.$

$= -2(\omega+\omega^2)$

$= -2 \cdot -1$

$= -2\times -1$

$= 2$

IMP

(iv) $(1-\omega+\omega^2)\times(1+\omega-\omega^2)=4$

$\cancel{= (\omega^2+\omega^2)}$

$=(1+\omega^2-\omega)(1+\omega-\omega^2)$

$=(-\omega-\omega)(-\omega^2-\omega^2)$

$=(-2\omega)(-2\omega^2)$

$=4\,\omega.\omega^2$

$=4\omega^3$

$=4\times 1$

$=4.$

(ii) $(2+5\omega+2\omega^2)^6$

$=(2+2\omega+3\omega+2\omega^2)^6$

$=(2+2\omega+2\omega^2+3\omega)^6$

$=\left[2(1+\omega+\omega^2)+3\omega\right]^6$

$=\left[3\omega\right]^6$

$=729\omega^6$

$=729(\omega^3)^2$

$=729(1)^2$

$=729$

$(2+2\omega+5\omega^2)^6$

$=(2+2\omega+2\omega^2+3\omega^2)^6$

$=\left[2(1+\omega+\omega^2)+3\omega^2\right]^6$

$=\left[3\omega^2\right]^6$

$=729\omega^{12}$

$=729(\omega^3)^4$

$=729(1)^4$

$=729$

$81 \times 3 = 243$, $243 \times 3 = 729$

(iv) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)=9$

$= (1-\omega)(1-\omega^2)(1-\omega^3\cdot\omega)(1-(\omega^3)^2\cdot\omega^2)$

$= (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$= (1-\omega)^2(1-\omega^2)^2$

$= [(1-\omega)(1-\omega^2)]^2$

$= [1-\omega^2-\omega+\omega^3]^2$

$= [1-\omega^2-\omega+1]^2$

$= [2-(\omega^2+\omega)]^2$

$= [2+1]^2$

$= (3)^2$

$= 9$

(v) $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})$

$= (2-\omega)(2-\omega^2)(2-(\omega^3)^3\cdot\omega)(2-(\omega^3)^3\cdot\omega^2)$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$= (2-\omega)^2(2-\omega^2)^2$

$= [(2-\omega)(2-\omega^2)]^2$

$= [4-2\omega^2-2\omega+\omega^3]^2$

~~$= [4-\omega-\omega^2]^2$~~

$= \cancel{4}$

$= [4-2\omega^2+2\omega+\omega^3]^2$

$= [4-2\omega^2+2\omega+1]^2$

$= [5-2(\omega^2+\omega)]^2$

$= [5-2(-1)]^2$

$= [5+2]^2$

$= (7)^2$

$= 49$

(vi) $(1+5\omega^2+\omega^4)(1+5\omega+\omega^2)(5+\omega+\omega^2)=64$

$= (1+5\omega^2+\omega^4)(1+5\omega+\omega^2)(5+\omega+\omega^2)$

$= (1+4\omega^2+\omega^2+\omega^4)(1+4\omega+\omega+\omega^2)(\cancel{5}\ 1+4+\omega+\omega^2)$

$= (1+4\omega^2+\omega^2+\omega^3\cdot\omega)(1+4\omega+\omega+\omega^2)(1+4+\omega+\omega^2)$

$= (1+\omega^2+\omega+4\omega^2)(1+\omega+\omega^2+4\omega)(1+\omega+\omega^2+4)$

$= (4\omega^2)(4\omega)(4)$

$= 64(\omega^3)$

$= 64\times 1$

$= 64$

IMP

Square Root of a complex No.

let, $z = a+ib$

$\sqrt{a+ib} = x+iy$

by squaring both sides :-

$\Rightarrow a+ib = (x+iy)^2$

$\Rightarrow a+ib = x^2 + i^2y^2 + 2x \cdot iy$

$\Rightarrow a+ib = x^2 + (-1)y^2 + i2xy$

$\Rightarrow a+ib = x^2 - y^2 + i\,2xy$

$x^2 - y^2 = a \qquad 2xy = b$

$$x = \pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} \qquad x = \pm\sqrt{\frac{Re(z)+|z|}{2}}$$

$$y = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}} \qquad y = \pm\sqrt{\frac{|z|-Re(z)}{z}}$$

*find a square root of $3+4i$

Soln let, $\sqrt{3+4i} = x+iy$

Squaring both sides :-

$3+4i = (x+iy)^2$

$3+4i = x^2 + i^2y^2 + i2xy$

$3+4i = x^2 + (-1)y^2 + i\,2xy$

$3+4i = x^2 - y^2 + i\,2xy$

on comparing both sides we get

$x^2 - y^2 = 3$

$2xy = 4$

we know $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= 3^2 + 4^2 = 9+16$

$= 25$

$x^2+y^2 = \sqrt{25} = 5$ ——— (2)

adding equation (1) and (2)

$$x^2-y^2=3$$
$$x^2+y^2=5$$
$$2x^2=8$$
$$x^2=\frac{8}{2}$$
$$x^2=4$$
$$x=\sqrt{4}=\pm 2$$

$$x^2+y^2=5$$
$$2^2+y^2=5$$
$$4+y^2=5$$
$$y^2=5-4$$
$$y^2=1$$
$$y=\sqrt{1}$$
$$y=\pm 1$$

$\therefore \sqrt{3+4i} = x+iy = 2 \pm i$

Hence,

Q) Find a square root of $-5+12\sqrt{-1}$.

$$x+iy = \sqrt{-5+12i} \qquad (\sqrt{-1}=i)$$

$$\Rightarrow \sqrt{-5+12i} = x+iy$$
$$\Rightarrow -5+12i = (x+iy)^2$$
$$\Rightarrow -5+12i = (x)^2 + i^2y^2 + 2.x.iy$$
$$\Rightarrow -5+12i = x^2+(-1)y^2 + i2xy.$$
$$\Rightarrow -5+12i = x^2-y^2+i2xy.$$

comparing both the side :-

$x^2 - y^2 = -5$ — (1)

$2xy = 12$ — (2).

we know!

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= (-5)^2 + (12)^2$

$= 25 + 144$

$= 169$

$(x^2+y^2)^2 = 169$

$x^2+y^2 = \sqrt{169} = \pm 13$.

adding equation (1) and (2).

$$\begin{array}{r} x^2 - y^2 = -5 \\ x^2 + y^2 = 13 \\ \hline 2x^2 = 18 \end{array}$$

$x^2 = \cancel{\frac{18}{2}}\ 9$

$x = \sqrt{9}$

$x = \pm 3$.

$x^2 - y^2 = 5$

$(3)^2 - y^2 = 5$

$9 - y^2 = 5$

$-y^2 = 5-9$

$+y^2 = +4$

$y = \sqrt{4} = \pm 2$

$\therefore \sqrt{-5+12i} = x+iy = 3 \pm 2i$

H.W

(a) $-15-8i$

(b) $-8+\sqrt{-1}$

(c) $-i$

(d) $1+4\sqrt{3}i$

$$\begin{array}{r} 225 \\ +64 \\ \hline 289 \end{array}$$

(a) $-15-8i$

$\Rightarrow \sqrt{-15-8i} = x+iy$

$\Rightarrow -15-8i = (x+iy)^2$

$\Rightarrow -15-8i = (x)^2+(iy)^2+2\cdot x\cdot iy$

$\Rightarrow -15-8i = x^2+i^2y^2+i2xy$

$\Rightarrow -15-8i = x^2+(-1)y^2+i2xy$

$\Rightarrow -15-8i = x^2-y^2+i2xy$.

$x^2-y^2=-15$ $\quad$ $2xy=-8$

$x^2-y^2=-15$ ——— (1)

$2xy=-8$ ——— (2)

$(x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$

$=(-15)^2+(-8)^2$

$=225+64$

$=289$

$(x^2+y^2)^2=289$

$x^2+y^2=\sqrt{289}=17$ ——— (3)

adding equation (1) and (3)

$$\begin{array}{r} x^2-y^2=-15 \\ x^2+y^2=17 \\ \hline 2x^2=2 \end{array}$$

$x^2 = \cancel{\ }\ 2/2=1$

$x=\cancel{\sqrt{-6}}\ 1$

$x^2-y^2=-15$

$\Rightarrow 1^2-y^2=-15$

$\Rightarrow -y^2=-15-1$

$\Rightarrow +y^2=+16$

$\Rightarrow y=\sqrt{16}$

$\Rightarrow y=4$

$\cancel{x^2-y^2=-15}$

$\cancel{(\sqrt{-6})^2-y^2=-15}$

$\cancel{-6-y^2=-15}$

$\cancel{-y^2=-15+6}$

$\cancel{-y^2=-9}$

$\cancel{y=\sqrt{9}}$

$\cancel{=\pm 3}$

~~$\sqrt{-15-8i}=$~~

$\therefore \sqrt{-15-8i}=x+iy$

$=1+4i$

(b) $-8+\sqrt{-1}$

$= -8-1i$

$\Rightarrow \sqrt{-8-i} = x+iy$

$\Rightarrow -8-i = (x+iy)^2$

$\Rightarrow -8-i = x^2+(iy)^2+2.x.iy$

$\Rightarrow -8-i = x^2+i^2y^2+i2xy$

$\Rightarrow -8-i = x^2+(-1)y^2+i2xy$

$\Rightarrow -8-i = x^2-y^2+i2xy.$

$x^2-y^2=-8 \qquad 2xy=-1$

$x^2-y^2=-8$ ———— (1)

$2xy=-1$ ———— (2)

$\cancel{x^2-y^2=-8}$

$(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2$

$=(-8)^2+(-1)^2$

$=64+1$

$=65$

$\cancel{(x^2+y^2)^2=65}$

$\cancel{x^2+y^2=\sqrt{65}}$

$(x^2+y^2)^2=65$

$\Rightarrow x^2+y^2=\sqrt{65}$

$x^2+y^2=\sqrt{65}$

$x^2-y^2=-8$

$2x^2=\sqrt{65}-8.$

$x^2=\dfrac{\sqrt{65}-8}{2}$

$x=\sqrt{\dfrac{\sqrt{65}-8}{2}}$

$x^2+y^2=\sqrt{65}$

$(-)\ x^2-y^2=-8$

$2y^2=\sqrt{65}+8$

$y^2=\dfrac{\sqrt{65}+8}{2}$

$y=\sqrt{\dfrac{\sqrt{65}+8}{2}}$

(c) $-i$

$= 0 - i$

$\sqrt{0-i} = x + iy$

$\Rightarrow -i = (x+iy)^2$

$\Rightarrow -i = (x)^2 + (iy)^2 + 2 \cdot x \cdot iy$

$\Rightarrow -i = x^2 + i^2y^2 + i2xy$

$\Rightarrow -i = x^2 + (-1)y^2 + i2xy$

$\Rightarrow -i = x^2 - y^2 + i2xy$

$x^2 - y^2 = 0$ — (1) $\quad 2xy = -1$ — (2)

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= 0 + (-1)^2$

$= (-1)^2$

$= 1.$

$(x^2+y^2)^2 = 1$

$x^2+y^2 = \sqrt{1} = \pm 1.$

$x^2 - y^2 = 0$

$x^2 + y^2 = 1$

$2x^2 = 1$

$x^2 = 1/2$

$x = \sqrt{1/2}$

$x^2 - y^2 = 0.$

$\Rightarrow (\sqrt{1/2})^2 - y^2 = 0$

$\Rightarrow 1/2 - y^2 = 0$

$\Rightarrow +y^2 = +1/2$

$\Rightarrow y = \sqrt{1/2}$

$\therefore \sqrt{-i} = x + iy = \sqrt{1/2} + i\sqrt{1/2}$

(d) $1+4\sqrt{3}i$

$\Rightarrow \sqrt{1+4\sqrt{3}i} = x+iy$

$\Rightarrow 1+4\sqrt{3}i = (x+iy)^2$

$\Rightarrow 1+4\sqrt{3}i = (x)^2+(iy)^2+2\cdot x\cdot iy$

$\Rightarrow 1+4\sqrt{3}i = x^2+i^2y^2+i2xy$

$\Rightarrow 1+4\sqrt{3}i = x^2+(-1)y^2+i2xy$

$\Rightarrow 1+4\sqrt{3}i = x^2-y^2+i2xy$

$x^2-y^2=1$ ——— (1) $\quad 2xy=4\sqrt{3}$ ——— (2)

$(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2$

$=(1)^2+(4\sqrt{3})^2$

$=1+16\times3$

$=1+48$

$=49$

$(x^2+y^2)^2=49$

$x^2+y^2=\sqrt{49}=\pm7$

$x^2-y^2=1$

$x^2+y^2=7$

$2x^2=8$

$x^2=8/2=4$

$x=\sqrt{4}$

$x=\pm2$

$x^2-y^2=1$

$2^2-y^2=1$

$4-y^2=1$

$-y^2=1-4$

$\cancel{-}y^2=\cancel{-}3$

$y=\sqrt{3}$

$y=\sqrt{3}$

$\therefore \sqrt{1+4\sqrt{3}i}=x+iy=2+\sqrt{3}i$

Q) Find the square root of a complex number $-8-6i$.

$\sqrt{-8-6i} = x+iy$

$\Rightarrow -8-6i = (x+iy)^2$

$\Rightarrow -8-6i = (x)^2 + 2x \cdot iy + (iy)^2$

$\Rightarrow -8-6i = x^2 + 2ixy + i^2y^2$

$\Rightarrow -8-6i = x^2 + i\,2xy + (-1)y^2$

$\Rightarrow -8-6i = x^2 + i\,2xy - y^2$

$\Rightarrow -8-6i = x^2 - y^2 + i\,2xy$

$x^2 - y^2 = -8$

$2xy = -6.$

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= (-8)^2 + (-6)^2$

$= 64 + 36$

$= 100$

$(x^2+y^2)^2 = 100$

$x^2+y^2 = \sqrt{100} = 10$

$x^2+y^2 = 10$

$x^2-y^2 = -8$

$2x^2 = 2$

$x^2 = \cancel{2}/\cancel{2} = 1$

$x = \sqrt{1} = \pm 1$

$x^2 - y^2 = -8$

$1^2 - y^2 = -8$

$1 - y^2 = -8$

$-y^2 = -8-1$

$y^2 = 9$

$y = \sqrt{9}$

$y = 3$

$y = \pm 3$

$\sqrt{-8-6i} =$

$\pm 1 + 3i$

$z = 1+\sqrt{3}\,i$

Amplitude/Argument (θ)

$$= \tan^{-1}\left(\frac{b}{a}\right)$$
$$= \tan^{-1}\frac{\sqrt{3}}{1}$$
$$= \tan^{-1}\sqrt{3}$$
$$= \frac{\pi}{3}$$

* Find the multiplicative inverse of a complex no.

$$z = (2+\sqrt{3}i)^2$$

$$= (2)^2 + (\sqrt{3}i)^2 + 2\cdot 2\cdot \sqrt{3}i$$
$$= 4 + 3i^2 + 4\sqrt{3}i$$
$$= 4 + 3(-1) + 4\sqrt{3}i$$
$$= 4 - 3 + 4\sqrt{3}i$$
$$= 1 + 4\sqrt{3}\,i$$

$$\frac{1}{z} = \frac{1}{(1+4\sqrt{3}i)}$$
$$= \frac{1\,(1-4\sqrt{3}i)}{(1+4\sqrt{3}i)(1-4\sqrt{3}i)}$$
$$= \frac{1-4\sqrt{3}i}{(1)^2-(4\sqrt{3}i)^2}$$
$$= \frac{1-4\sqrt{3}i}{1-48i^2}$$
$$= \frac{1-4\sqrt{3}i}{1-48(-1)}$$
$$= \frac{1-4\sqrt{3}i}{1+48}$$
$$= \frac{1-4\sqrt{3}i}{49}$$

Find the real part of x and y :—

$(x+iy)(2-3i) = 4+i$

$\Rightarrow x+iy = \dfrac{4+i\,(2+3i)}{(2-3i)(2+3i)}$

$\Rightarrow x+iy = \dfrac{8+12i+2i+3i^2}{(2)^2-(3i)^2}$

$\Rightarrow x+iy = \dfrac{8+14i+3(-1)}{4-9i^2}$

$\Rightarrow x+iy = \dfrac{8+14i-3}{4-9(-1)}$

$\Rightarrow x+iy = \dfrac{8-3+14i}{4+9}$

$\Rightarrow x+iy = \dfrac{5+14i}{13}$

$x = \dfrac{5}{13} \quad y = +\dfrac{14}{13}i$

IMP

Demovier's Theorem

(i) If 'n' is an integer
Then 'n' is zero or positive or Negative
The statement is $(\cos\theta + i\sin\theta)^n$
$= \cos n\theta + i\sin n\theta$.

(ii) If 'n' is a fraction or Natural no. Then the Demovier's theorem is $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

Note:-

(1) $\cos n\theta + i\sin n\theta \neq \cos n\theta - i\sin n\theta$

(2) $\cos 0 + i\sin 0 = 1$ $(n=0)$

(3) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$ $(n=1, \theta=\pi/2)$

$\cos n\theta + i\sin n\theta$

$= \cos\pi/2 + i\sin\frac{\pi}{2}$

$= 0 + i\times 1$

$= i$

(4) $\cos\pi + i\sin\pi = -1$ $(n=1, \theta=\pi)$

$\cos\pi + i\sin\pi$

$= -1 + i\times 0$

$= -1$

(5) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$

IMP * If $x + \frac{1}{x} = 2\cos\theta$ then show that

(a) $x^n + \frac{1}{x^n} = 2\cos n\theta$. (b) $x^n - \frac{1}{x^n} = 2 i\sin n\theta$

$x + \frac{1}{x} = 2\cos\theta$.

$\Rightarrow \frac{x^2+1}{x} = 2\cos\theta$

$\Rightarrow x^2+1 = 2x\cos\theta$ $\{\sin^2\theta + \cos^2\theta = 1\}$

$\Rightarrow x^2 + \sin^2\theta + \cos^2\theta = 2x\cos\theta$

$\Rightarrow x^2 - 2x\cos\theta + \cos^2\theta = -\sin^2\theta$. $[\because i^2 = -1]$

$\Rightarrow (x - \cos\theta)^2 = i^2\sin^2\theta = (i\sin\theta)^2$

$\Rightarrow x - \cos\theta = i\sin\theta$.

$x = i\sin\theta + \cos\theta$

$x = \cos\theta + i\sin\theta$.

By Demovier's The^m

$$x^n = (\cos\theta + i\sin\theta)^n$$
$$= \cos n\theta + i\sin n\theta.$$

$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$

$$\frac{1}{x^n} = \frac{1}{(\cos\theta + i\sin\theta)^n} = (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta.$$

$$\therefore x^n + \frac{1}{x^n}.$$
$$= \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}$$
$$= 2\cos n\theta \text{ (proved)}$$

(ii)

$$x + \frac{1}{x} = 2\cos\theta$$
$$\Rightarrow \frac{x^2+1}{x} = 2\cos\theta.$$
$$\Rightarrow x^2 + 1 = 2x\cos\theta.$$
$$\Rightarrow x^2 + \sin^2\theta + \cos^2\theta = 2x\cos\theta$$
$$\Rightarrow x^2 - 2x\cos\theta + \cos^2\theta = -\sin^2\theta$$
$$\Rightarrow (x - \cos\theta)^2 = i^2\sin^2\theta$$
$$\Rightarrow (x - \cos\theta)^2 = (i\sin\theta)^2$$
$$\Rightarrow x - \cos\theta = i\sin\theta$$
$$\Rightarrow x = \cos\theta + i\sin\theta.$$

By Demovier's the^m

$$x^n = (\cos\theta + i\sin\theta)^n$$
$$= \cos n\theta + i\sin n\theta$$
$$\frac{1}{x^n} = \frac{1}{(\cos\theta + i\sin\theta)^n} = (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta.$$

$x^n - \frac{1}{x^n}$

$= (\cos n\theta + i \sin n\theta) - (\cos n\theta - i \sin n\theta)$

$= \cos n\theta + i \sin n\theta - \cos n\theta + i \sin n\theta$

$= 2i \sin n\theta$

UNIT-1 Matrices.

It is the arrangement of numbers by row and column in a rectangular array.

e.g.- $$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$m \rightarrow$ Row
$n \rightarrow$ column

⇒ It is denoted by $[a_{ij}]$ or (a_{ij}).
⇒ It also denoted by capital Alphabate.

Types of matrix

Square Matrix :- Number of Row equal to number of Column.

Let, A be a square matrix. If it has same number of row and columns.

e.g.- $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}_{2\times2}$ → leading / main diagonal.

- **Lower triangular Matrix** :- A Square matrix is said to be Lower triangular matrix, if all the elements above the main diagonal are zero.

Ex :-
$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}_{3\times3}$$ — lower triangular matrix.

- **Upper triangular Matrix** :- A Square matrix is said to be upper triangular matrix, if all the elements below the main diagonal are zero.

Ex :
$$B = \begin{bmatrix} 2 & 5 & -6 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}_{3\times3}$$ — Upper triangular matrix.

~~Hermitian~~ Symmetric Matrix - A square matrix is called ~~hermitian~~ Symmetric matrix if it satisfies $\boxed{A^T = A}$

e.g. $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$ $\quad \boxed{\therefore A^T = A}$

$A^T = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$

e.g. $B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -2 & 4 \\ -3 & 4 & 3 \end{bmatrix}$

$B^T = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -2 & 4 \\ -3 & 4 & 3 \end{bmatrix}$ $\quad \boxed{B^T = B}$

Skew Symmetric Matrix - A square matrix is called skew symmetric matrix if it satisfies $\boxed{\therefore A^T = -A}$

e.g. - $A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$

$-A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ $\quad \boxed{\therefore A^T = -A}$

Rank of a matrix

Rank of a matrix is obtained by row operation, then the number of non-zero rows gives the rank of a matrix

IMP

e.g. :- find the rank of a matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

Sol^n

Let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix}_{3\times 3}$ $R_3 \to R_3 - 2\times R_2$, $R_3 \to R_3 - 2R_2$

$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, rank of matrix $A = 2$.

⇒ The rank of a matrix is denoted by $r(A)$ or $\rho(A)$, where 'A' is a matrix.

⇒ Rank of a matrix i.e. $r(A) \le$ (number of row, number of column)

i.e. $r(A) \le \min(m, n)$

⇒ Rank of A = Rank of A^T.

H.W

Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1 \times 2$

$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, rank of matrix $A = 2$.

Q) Find the rank of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

let,

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_2 + 2 \times R_1 \\ R_3 \longrightarrow R_3 + R_1 \end{matrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \quad R_3 \longrightarrow 2 \times R_3 - R_2$$

$$= \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2.$$

Q) Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

let,

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - 2 \times R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \quad R_3 \longrightarrow R_3 - R_2$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(B) = 2.$$

a) Find the rank of matrix $\begin{bmatrix} 1 & 3 & 2 & 5 & 1 \\ 2 & 2 & -1 & 6 & 3 \\ 1 & 1 & 2 & 3 & -1 \\ 0 & 2 & 5 & 2 & -3 \end{bmatrix}$

let,

$$A = \begin{bmatrix} 1 & 3 & 2 & 5 & 1 \\ 2 & 2 & -1 & 6 & 3 \\ 1 & 1 & 2 & 3 & -1 \\ 0 & 2 & 5 & 2 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \longrightarrow R_2 - 2\times R_1 \\ R_3 \longrightarrow R_3 - R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 3 & 2 & 5 & 1 \\ 0 & -4 & -5 & -4 & 1 \\ 0 & -2 & 0 & -2 & -2 \\ 0 & 2 & 5 & 2 & -3 \end{bmatrix} \quad \begin{array}{l} R_3 \longrightarrow 2\times R_3 - R_2 \\ R_4 \longrightarrow \cancel{2\times R_4 + R_2} \\ \qquad R_4 + R_3 \end{array}$$

$$= \begin{bmatrix} 1 & 3 & 2 & 5 & 1 \\ 0 & -4 & -5 & -4 & 1 \\ \cancel{0} & \cancel{2} & \cancel{0} & \cancel{2} & \cancel{2} \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 5 & 0 & -5 \end{bmatrix} \quad R_4 \longrightarrow R_4 - R_3$$

$$= \begin{bmatrix} 1 & 3 & 2 & 5 & 1 \\ 0 & -4 & -5 & -4 & 1 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 3$.

2) Find the rank of matrix = $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

Let,

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_2 - 2\times R_1 \\ R_3 \longrightarrow R_3 - 3\times R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & 6 \end{bmatrix} \quad R_3 \longrightarrow R_3 - R_2$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2.$

H.W.

2) Find the rank of matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_2 + R_1 \\ R_3 \longrightarrow R_3 - 2\times R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 1$

Q) Find the rank of the matrix $= A = \begin{bmatrix} -1 & -2 & 3 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

let,

$$A = \begin{bmatrix} -1 & -2 & 3 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_2 + 6 \times R_1 \\ R_3 \longrightarrow R_3 + 5 \times R_1 \end{matrix}$$

$$= \begin{bmatrix} -1 & -2 & 3 \\ 0 & 0 & 24 \\ 0 & 0 & 20 \end{bmatrix}$$

Date- 9/08/23 $\quad \therefore \rho(A) = 3$

2) $\begin{bmatrix} 1 & 2 & 3 \\ 9 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 2 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 2 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_2 - 9 \times R_1 \\ R_3 \longrightarrow R_3 - 7 \times R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -13 & -21 \\ 0 & -6 & -12 \\ 0 & 2 & 1 \end{bmatrix} \quad \begin{matrix} \cancel{R_3 \longrightarrow R_3 + 2 \times R_1} \\ R_4 \longrightarrow 3R_4 + R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -13 & -21 \\ 0 & -6 & -12 \\ 0 & 0 & -9 \end{bmatrix}$$

$$\therefore \rho(A) = 4$$

~~Soln.~~

System of linear equation

Homogenious | Non-homogeneous

$ax + by + cz = 0$ | $ax + by + cz = d$

e.g. $x + 2y - 3z = 0$ | e.g. $2x - 3y + 4z = 5$

$$a_1x + b_1y + c_1z = d_1 \quad \text{--- (1)}$$
$$a_2x + b_2y + c_2z = d_2 \quad \text{--- (2)}$$
$$a_3x + b_3y + c_3z = d_3 \quad \text{--- (3)}$$

The given equation written in matrix form by $\boxed{AX = B}$.

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ — co-efficient matrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Augmented Matrix (K)

$$K = [A \,\vdots\, B]$$

$$K = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$A \rightarrow$ coefficient matrix

$K \rightarrow$ Augmented matrix

→ (1) If rank of (A) = rank of (K) = Number of unknown.
→ System of linear eq^n is consistent and have a unique solution.

→ (2) If rank of (A) = rank of (K) < No. of unknown
→ equation is consistent but it has an infinite solution.

(3) If rank of (A) ≠ r(K)
→ equation is inconsistent and No-solution.

Solve,

$x+2y-z=3$

$3x-y+2z=1$

$2x-2y+3z=2$

The given equation can be written by matrix form $AX=B$.

$$A=\begin{bmatrix}1 & 2 & -1\\ 3 & -1 & 2\\ 2 & -2 & 3\end{bmatrix} \quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \quad B=\begin{bmatrix}3\\ 1\\ 2\end{bmatrix}$$

$$K=[A \mid B]=\left[\begin{array}{ccc|c}1 & 2 & -1 & 3\\ 3 & -1 & 2 & 1\\ 2 & -2 & 3 & 2\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}1 & 2 & -1 & 3\\ 3 & -1 & 2 & 1\\ 2 & -2 & 3 & 2\end{array}\right] \quad \begin{array}{l}R_2 \longrightarrow R_2-3\times R_1\\ R_3 \longrightarrow R_3-2\times R_1\end{array}$$

$$=\left[\begin{array}{ccc|c}1 & 2 & -1 & 3\\ 0 & -7 & 5 & -8\\ 0 & -6 & 5 & -4\end{array}\right] \quad R_3 \longrightarrow R_3-R_2$$

$$=\left[\begin{array}{ccc|c}1 & 2 & -1 & 3\\ 0 & -7 & 5 & -8\\ 0 & 1 & 0 & 4\end{array}\right]$$

$\therefore r(A)=3$

$r(K)=3$ $\Big] = $ (No. of unknown)

$\therefore$ Hence, the system of equation is consistent & It has a unique sol^n.

Sol^n

$x + 2y - z = 3$ ——— ①

$-7y + 5z = -8$ ——— ②

$y = 4$ ——— ③

$-7y + 5z = -8$

$-7\times4 + 5z = -8$

$-28 + 5z = -8$

$5z = -8 + 28$

$5z = 20$

$z = \frac{\cancel{20}\,4}{\cancel{5}}$

$z = 4$

~~$7y+5z=$~~

$x + 2y - z = 3$

$x + 2\times4 - 4 = 3$

$x + 8 - 4 = 3$

$x + 4 = 3$

$x = 3 - 4$

$x = -1$

∴ Hence $x = -1, y = 4$ and $z = 4$.

H.W

⑧ Test for consistency and solve $5x + 3y + 7z = 4, 3x + 26y + 2z = 9, 7x + 2y + 10z = 5$

⑨ Test for consistency and solve $2x - 3y + 7z = 5, 3x + y - 3z = 13, 2x + 19y - 47z = 32$

⑧ —

$5x + 3y + 7z = 4$
$3x + 26y + 2z = 9$
$7x + 2y + 10z = 5$

Here,

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

* check wether the following system of equation is consistent Solve if it is consistent.

$4x + 3y + 2z = -7$, $2x + y - 4z = -1$, $x + 2y + z = 1$

$$\begin{aligned} 4x + 3y + 2z &= -7 \\ 2x + y - 4z &= -1 \\ x + 2y + z &= 1. \end{aligned}$$

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 1 & -4 \\ 1 & 2 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} -7 \\ -1 \\ 1 \end{bmatrix}$$

$$K = \left[\begin{array}{ccc|c} 4 & 3 & 2 & -7 \\ 2 & 1 & -4 & -1 \\ 1 & 2 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \longrightarrow 2R_2 - R_1 \\ R_3 \longrightarrow 4 \times R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 4 & 3 & 2 & -7 \\ 0 & -1 & -10 & -9 \\ 0 & 5 & 2 & -3 \end{array}\right] \quad R_3 \longrightarrow R_3 + 5 \times R_2$$

$$= \left[\begin{array}{ccc|c} 4 & 3 & 2 & -7 \\ 0 & -1 & -10 & -9 \\ 0 & 0 & -48 & -48 \end{array}\right] \quad \cancel{R_3 \rightarrow}$$

OR

$$\Longrightarrow$$

$R(A) = 3$

$R(K) = 3$

$$= \left[\begin{array}{ccc|c} 4 & 3 & 2 & -7 \\ 2 & 1 & -4 & -1 \\ 1 & 2 & 1 & 1 \end{array}\right] \quad R_3 \longleftrightarrow R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 1 & -4 & -1 \\ 4 & 3 & 2 & -7 \end{array}\right] \quad \begin{array}{l} R_2 \longrightarrow R_2 - 2R_1 \\ R_3 \longrightarrow R_3 - 4R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -5 & -2 & -11 \end{array}\right] \quad R_3 \longrightarrow 3R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -12 & 0 & -30 \end{array}\right] \quad \begin{array}{l} R(A) = 3 \\ R(K) = 3, \end{array}$$

$\therefore$ equation is constitent and it has unique solution.

$$x + 2y + z = 1$$
$$-3y - 6z = -3$$
$$-12y = -30$$
$$y = \frac{30}{12} = \cancel{2.5} = \frac{10}{4} = \frac{5}{2}$$

$$12 \overline{)30} \, 2.5$$
$$\underline{24}$$
$$\underline{60}$$
$$0$$

put $y = 5/2$

$$-3y - 6z = -3$$
$$-3 \times 5/2 - 6z = -3$$
$$-15/2 - 6z = -3$$
$$-6z = -3 + 15/2 = \frac{-6+15}{2} = 9/2$$
$$z = 9/2 \times -6 = \frac{9}{-24} = -3/4$$

$x + 2y + z = 1$

$x + 2 \times 5/2 + -3/4 = 1 \qquad x + 5 - 3/4 = 1$

$x + \frac{5}{7} - \frac{3}{4} = 1$

$\Rightarrow x + \frac{20-3}{4} = 1$

$\Rightarrow x + \frac{17}{4} = 1$

$\Rightarrow x = \frac{1}{1} - \frac{17}{4}$

$\Rightarrow x = \frac{4-17}{4}$

$\Rightarrow x = -\frac{13}{4}$

Q) test the consistence,

$x - 5y + 3z = -1$

$2x - y - z = 5.$

$5x - 7y + z = 2$

Q) $A = \begin{bmatrix} 1 & -5 & 3 \\ 2 & -1 & -1 \\ 5 & -7 & 1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$

$K = \left[\begin{array}{ccc|c} 1 & -5 & 3 & -1 \\ 2 & -1 & -1 & 5 \\ 5 & -7 & 1 & 2 \end{array}\right]$ $R_2 \longrightarrow R_2 - 2 \times R_1$, $R_3 \longrightarrow R_3 - 5 \times R_1$

$= \left[\begin{array}{ccc|c} 1 & -5 & 3 & -1 \\ 0 & 9 & -7 & 7 \\ 0 & 18 & -14 & 7 \end{array}\right]$ $R_3 \longrightarrow R_3 - 2 \times R_2$

$= \left[\begin{array}{ccc|c} 1 & -5 & 3 & -1 \\ 0 & 9 & -7 & 7 \\ 0 & 0 & 0 & -7 \end{array}\right]$

$R(A) = 2$

$R(K) = 3$ $\therefore$ It is in consistent.

a) for what value of λ, does the system.

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$-x + 2y + z = 1$
$3x - y + 2z = 1$
$y + \lambda z = 1$

has (i) no solution (ii) unique solⁿ (iii) more than one solⁿ.

Solⁿ

$$K = \left[\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 1 & \lambda & 1 \end{array}\right] \quad R_2 \to R_2 + 3R_1$$

$$= \left[\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ 0 & 5 & 5 & 4 \\ 0 & 1 & \lambda & 1 \end{array}\right] \quad R_3 \to 5R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ 0 & 5 & 5 & 4 \\ 0 & 0 & 5\lambda-5 & 1 \end{array}\right]$$

(i) If $\lambda = 1$, then $5\lambda - 5 = 0$

Rank of $A = 2$
Rank of $k = 3$

$\therefore$ It is inconsistent so it has no solution.

(ii) If $\lambda \neq 1$, any number

Rank of $A = 3$
Rank of $k = 3$] = Number of known.

$\therefore$ It is consistent so it has unique value.

(iii). More than one solution

The system of linear equation does not satisfies the infinite number of solution.

Q) For what value of λ and μ do the system of equation

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$ have

(i) no solⁿ (ii) unique solⁿ (iii) infinite solⁿ.

Solⁿ the above equation can be written in matrix - form $AX=B$

$$A=\begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix} \quad B=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$$

$$K=\left[\begin{array}{ccc|c}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{array}\right] \quad \begin{array}{l}R_2\longrightarrow R_2-R_1\\R_3\longrightarrow R_3-R_1\end{array}$$

$$=\left[\begin{array}{ccc|c}1&1&1&6\\0&1&2&4\\0&1&\lambda-1&\mu-6\end{array}\right] \quad R_3\longrightarrow R_3-R_2$$

$$=\left[\begin{array}{ccc|c}1&1&1&6\\0&1&2&4\\0&0&\lambda-3&\mu-10\end{array}\right]$$

$\mu \neq 10$
$\lambda = 3$

(i) If $\lambda=3$, but $\mu\neq10$,

$\therefore$ Rank of $K=3$
Rank of $A=2$

So, it is inconsistent and it has no solution.

(ii) $\lambda\neq3$, any value.
$\mu\neq10$, any value,

Then, so Rank of $A=3$
Rank of $K=3$

So it is consistent and it has unique solution.

(iii) $\lambda = 3$
$\mu = 10$

Then, rank of $A = 2$
rank of $K = 2$

$\therefore$ It is consistent and it has infinite solution.

IMP

(Q) $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$;

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$K = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right] \quad R_1 \longrightarrow \tfrac{1}{2}R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 3/2 & 5/2 & 9/2 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right] \quad \begin{array}{l} R_2 \longrightarrow R_2 - 7R_1 \\ R_3 \longrightarrow R_3 - 2R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 3/2 & 5/2 & 9/2 \\ 0 & -15/2 & -39/2 & -47/2 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$$

$3 - \frac{3}{2} \times 2 = 0$

$\lambda - 2 \times 5/2 = \lambda - 5$

$\mu - 9/2 \times 2$

$\frac{3}{1} - \frac{21}{2} = \frac{6-21}{2} = -\frac{15}{2}$

$\frac{-2}{1} - \frac{35}{2} = \frac{-4-35}{2} = -\frac{39}{2}$

$\frac{8}{1} - \frac{63}{2} = \frac{16-63}{2} = \frac{-47}{2}$

(i) If $\lambda = 5$, $\mu \neq 9$.
Rank of $A = 2$
Rank of $K = 3$

So, it is inconsistent and it has no solution.

(ii) If $\lambda \neq 5$, $\mu \neq 9$
Then, Rank of $A = 3$
Rank of $K = 3$

$\therefore$ It is Consistent and it has unique solution.

$\lambda = 5$
$\mu = 9$
Rank of A = 2
Rank of K = 2
$\therefore$ It is consistent and it has infinite value

Date-16/08/23

Q/ Determine the value of λ then set of

$$3x_1 + x_2 - \lambda x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$$

$$K = \left[\begin{array}{ccc|c} 3 & 1 & -\lambda & 0 \\ 4 & -2 & -3 & 0 \\ 2\lambda & 4 & \lambda & 0 \end{array}\right] \quad R_1 \longrightarrow \tfrac{1}{3} \times R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 1/3 & -\lambda/3 & 0 \\ 4 & -2 & -3 & 0 \\ 2\lambda & 4 & \lambda & 0 \end{array}\right] \quad \begin{array}{l} R_2 \longrightarrow R_2 - 4R_1 \\ R_3 \longrightarrow R_3 - 2\lambda R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1/3 & \lambda/3 & 0 \\ 0 & -10/3 & -\frac{(9+4\lambda)}{3} & 0 \\ 0 & \frac{12-2\lambda}{3} & \frac{\lambda+2\lambda^2}{3} & 0 \end{array}\right]$$

Q/ $x+y+z=6$

$x+2y+3z=10$

$x+2y+\lambda z=\mu$

$$K=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \quad \begin{array}{l} R_2 \to R_2-R_1 \\ R_3 \to R_3-R_1 \end{array}$$

$$=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \quad R_3 \to R_3-R_2$$

$$=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

(i) $\therefore$ $\lambda \neq 3$ or $\mu=10$
$\mu \neq 10$

$R(A) = 3 = R(K)$ (No. of unknown)

$\therefore$ consistent & unique solution.

(ii) $\lambda=3$

$\mu \neq 10$, $R(A)=2$
$R(K)=3$

$R(A) \neq R(K)$

$\therefore$ It is not consistent.

(iii) $\lambda=3$

$\mu=10$ $R(A)=2$
$R(K)=2$

$R(A)=R(K)$ (not equal to unknown)

$\therefore$ consistent & infinite solution.

Rouch's Theorem :-

If the system of equation is consistent of the rank of co-efficient matrix (A) is equal to the rank of augmented matrix (K) otherwise the system of equation is inconsistent.

Rank of A = Rank of K (consistent)
Rank of A ≠ Rank of K (Inconsistent)

* Determine the value of 'λ' for which the system of equation may posses non-trivial soln.

$$3x_1 + x_2 - \lambda x_3 = 0,\quad 4x_1 - 2x_2 - 3x_3 = 0$$
$$2\lambda x_1 + 4x_2 + \lambda x_3 = 0.$$

Soln

$$K = \left[\begin{array}{ccc|c} 3 & 1 & -\lambda & 0 \\ 4 & -2 & -3 & 0 \\ 2\lambda & 4 & \lambda & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & -\lambda & 0 \\ -2 & 4 & -3 & 0 \\ 4 & 2\lambda & \lambda & 0 \end{array}\right] \quad \begin{array}{l} R_2 \longrightarrow R_2 + 2R_1 \\ R_3 \longrightarrow R_3 - 4R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & -\lambda & 0 \\ 0 & 10 & -3-2\lambda & 0 \\ 0 & 2\lambda-12 & 5\lambda & 0 \end{array}\right]$$

$$x \Rightarrow x + 3y - \lambda z = 0$$
$$\Rightarrow 10y + (-3-2\lambda)z = 0$$
$$\Rightarrow (2\lambda - 12)y + 5\lambda z = 0$$

$$\Rightarrow 50\lambda - [(-3-2\lambda)(2\lambda-12)] = 0$$
$$\Rightarrow 50\lambda - [-6\lambda + 36 - 4\lambda^2 + 24\lambda] = 0$$
$$\Rightarrow 50\lambda + 6\lambda - 36 + 4\lambda^2 - 24\lambda = 0$$
$$\Rightarrow \frac{4\lambda^2}{4} + \frac{32\lambda}{4} - \frac{36}{4} = 0$$
$$\Rightarrow \lambda^2 + 8\lambda - 9 = 0$$
$$\Rightarrow \lambda^2 + 9\lambda - \lambda - 9 = 0$$
$$\Rightarrow \lambda(\lambda + 9) - 1(\lambda + 9) = 0$$
$$\Rightarrow (\lambda + 9)(\lambda - 1)$$

$\lambda+9=0$, $\lambda-1=0$
$\lambda=-9$, $\lambda=1$.

$x+3y-z=0$ ——— (1)
$10y-5z=0$ ——— (2)
$-10y+5z=0$ ——— (3)

From Eqn (3)
$+10y=+5z$
$y=\frac{5}{10}z=\frac{1}{2}z$
$\frac{y}{1}=\frac{z}{2}=\lambda$

$y=\lambda,\ z=2\lambda$
From equation (1) :-
$x+3y-z=0$
$\Rightarrow x+3\times\lambda-2\lambda=0$
$\Rightarrow x+3\lambda-2\lambda=0$
$\Rightarrow x+\lambda=0$
$\Rightarrow \boxed{x=-\lambda}$

$x=-1$
$y=1$
$z=2$

H.W
1 - Rank of matrix
Rank of matrix is the largest order of the non-vanishing minor of that matrix.

2) $5x+3y+7z=4$
$3x+26y+2z=9$
$7x+2y+10z=5$

$$A=\begin{bmatrix}5&3&7\\3&26&2\\7&2&10\end{bmatrix} \quad B=\begin{bmatrix}4\\9\\5\end{bmatrix}. \quad x=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$K=\left[\begin{array}{ccc|c}5&3&7&4\\3&26&2&9\\7&2&10&5\end{array}\right] \quad R_1\rightarrow \tfrac{1}{5}\times R_1$$

$$=\left[\begin{array}{ccc|c}1&3/5&7/5&4/5\\3&26&2&9\\7&2&10&5\end{array}\right] \quad \begin{array}{l}R_2\rightarrow R_2-3R_1\\R_3\rightarrow R_3-7R_1\end{array}$$

$$=\left[\begin{array}{ccc|c}1&3/5&7/5&4/5\\0&121/5&-11/5&33/5\\0&-11/5&1/5&-3/5\end{array}\right] \quad R_3\rightarrow 11\times R_3+R_2$$

$$=\left[\begin{array}{ccc|c}1&3/5&7/5&4/5\\0&12/5&-11/5&33/5\\0&0&0&0\end{array}\right]$$

$R(A)=2$
$R(K)=2.$

$\therefore$ It is consistent but it has infinite number of solutions.

$$x+\tfrac{3}{5}y+\tfrac{7}{5}z=\tfrac{4}{5}$$
$$\tfrac{12}{5}y-\tfrac{11}{5}z=\tfrac{33}{5}$$

$\frac{26}{1}-\frac{9}{5}$
$=\frac{130-9}{5}$
$=\frac{121}{5}$

$\frac{2}{1}-\frac{21}{5}$
$=\frac{10-21}{5}$
$=\frac{-11}{5}$

$\frac{9}{1}-\frac{12}{5}$
$=\frac{45-12}{5}$
$=\frac{33}{5}$

$\cancel{\frac{2}{1}-\frac{21}{5}}$
$=\cancel{\frac{10-21}{5}}$

$\frac{10}{1}-\frac{49}{5}$
$=\frac{50-49}{5}$
$=\frac{1}{5}$

$\frac{5}{1}-\frac{28}{5}$
$=\frac{25-28}{5}=-\frac{3}{5}$

3) Define lower triangular matrix with an example.

→ A square matrix is said to be lower triangular matrix, if all the elements above the main diagonal are zero.

$$Ex - \begin{bmatrix} -1 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 5 & 3 \end{bmatrix}$$

4) Determine the rank of matrix.

$$\rightarrow \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} \quad R_1 \rightarrow R_1 \times 1/3.$$

$$= \begin{bmatrix} 1 & -1/3 & 2/3 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 + 6R_1 \\ R_3 \rightarrow R_3 + 3R_1 \end{matrix}$$

$2 - \frac{6^2}{3}$
$2-2$
$=0$
$4 - 6^2 \times \frac{2}{3}$
$= 4-4$
$=0$

$1 - 1/3 \times 3$
$=0$
$2 + 3 \times 2/3$
$= 4$

$$= \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Rank of $(A) = 2$.

(5) $4x+3y+2z=-7$

$2x+y-4z=-1$

$x+2y+z=1$

$$A=\begin{bmatrix}4&3&2\\2&1&-4\\1&2&1\end{bmatrix} \quad B=\begin{bmatrix}-7\\-1\\1\end{bmatrix} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}.$$

$$\cancel{A=[1\ 2\ 1]} \quad K=\left[\begin{array}{ccc|c}4&3&2&-7\\2&1&-4&-1\\1&2&1&1\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}1&2&1&1\\2&1&-4&-1\\4&3&2&-7\end{array}\right] \quad \begin{array}{l}R_2\longrightarrow R_2-2R_1\\R_3\longrightarrow R_3-4R_1\end{array}$$

$$=\left[\begin{array}{ccc|c}1&2&1&1\\0&-3&-6&-3\\0&-5&-2&-11\end{array}\right] \quad R_3\longrightarrow 3R_3-R_2$$

$$=\left[\begin{array}{ccc|c}1&2&1&1\\0&-3&-6&-3\\0&-12&0&-30\end{array}\right]$$

$$\left.\begin{array}{l}R(A)=3\\R(K)=3\end{array}\right] = \text{Number of unknown}$$

∴ It is consistent and it has unique solution.

$x+2y+z=1$

$-3y-6z=-3$

$-12y=-30$

$y=\frac{30}{12}=\frac{10}{4}=\frac{5}{2}.$

$-3y-6z=-3$

$-(3y+6z)=-3$

$3\times\frac{5}{2}+6z=3$

$\frac{15}{2}+6z=3$

$$6z = \frac{3}{1} - \frac{15}{2}$$

$$6z = \frac{6-15}{2}$$

$$6z = \frac{-9}{2}$$

$$z = \frac{-9}{2 \times 6}$$

$$z = \frac{\cancel{-9}\,3}{\cancel{12}\,4} = -3/4.$$

$$x + 2y + z = 1$$

$$x + \frac{\cancel{2} \times 5}{\cancel{2}} + -3/4 = 1$$

$$x + 5 - 3/4 = 1$$

$$x + 5 = \frac{1}{1} + \frac{3}{4}$$

$$x + 5 = \frac{4+3}{4}$$

$$x + 5 = 7/4$$

$$x = 7/4 - \frac{5}{1}$$

$$x = \frac{7-20}{4}$$

$$x = -13/4$$

(6) Find the rank of the matrix,

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$R(A) = 1$.

(7) $2x+3y+4z=2$
$3x+4y-2z=5$
$4x+6y+\lambda z=\mu$

$$A=\begin{bmatrix}2&3&4\\3&4&-2\\4&6&\lambda\end{bmatrix} \quad x=\begin{bmatrix}x\\y\\z\end{bmatrix} \quad B=\begin{bmatrix}2\\5\\\mu\end{bmatrix}$$

$$K=\left[\begin{array}{ccc|c}2&3&4&2\\3&4&-2&5\\4&6&\lambda&\mu\end{array}\right] \quad R_1\longrightarrow R_1\times 1/2$$

$$=\left[\begin{array}{ccc|c}1&3/2&4/2&2/2\\3&4&-2&5\\4&6&\lambda&\mu\end{array}\right] \quad \begin{array}{l}R_2\longrightarrow R_2-3R_1\\R_3\longrightarrow R_3-4R_1\end{array}$$

$$=\left[\begin{array}{ccc|c}1&3/2&2&1\\0&-1/2&-8&2\\0&0&\lambda-8&\mu-1\end{array}\right]$$

$4-\frac{3}{2}\times 3$
$4-\frac{9}{2}$
$=\frac{8-9}{2}$
$=-1/2$

$-2-3\times 4/2$
$=-2-\frac{12}{2}$
$=\frac{-4-12}{2}$
$=\frac{-16}{2}=8$

$5-\frac{2}{2}\times 3$
$\frac{5}{1}-\frac{6}{2}$
$=\frac{10-6}{2}$
$=4/2=2$

$6-3/2\times 4$
$6-6$
$=0$

$\lambda-4\times 4/2$
$=\lambda-16/2$
$=\lambda-8$

(i) For no soln.
$\lambda\neq 8$, but $\mu=1$.
Then, $R(A)=2$.
$R(K)=3$.
So there is no solution.

(ii) For infinite solution
$\lambda=8$
$\mu=1$, then.
$R(A)=2$.
$R(K)=2$
So infinite solution possible.

(iii) For unique solution.
$\lambda\neq 8$
$\mu\neq 1$ Then $R(A)=3$
$R(K)=3$
So there has unique solution.

(8)
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$
$$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$
$$R(A) = 1.$$

(10)
$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad R_2 \to R_2 + R_1$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad R_3 \to R_3 + R_2$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$
$$R(A) = 3.$$

(9)
$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$K = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

$\therefore$ No solution

$\lambda = 3$
$\mu \neq 10$

$\therefore$ Unique solution

$\lambda \neq 3$
$\mu \neq 10$

$\therefore$ Infinite solution

$\lambda = 3$
$\mu = 10$

(11)

$x + 2y - 3z = 2$
$3x - y - 2z = 1$
$2x + 3y - 5z = k$

$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & -2 \\ 2 & 3 & -5 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 1 \\ k \end{bmatrix}$

$K = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & k \end{array}\right]$ $R_2 \longrightarrow R_2 - 3R_1$
$R_3 \longrightarrow R_3 - 2R_1$

$-1 - 3 \times 2$
$-1 - 6$
$= -7$

$-2 + 9$
$= 7$

$1 - 6$
$= -5$

$3 - 4$
$= -1$

$-5 + 6$
$= -1$

$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -7 & 7 & -5 \\ 0 & -1 & -1 & k-4 \end{array}\right]$

$k \neq 4$

(12)

$A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ $R_2 \longrightarrow R_2 - 3R_1$

$= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ $\therefore R(A) = 1.$

(13)

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$= \begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 3 \\ 2 & 6 & 5 \end{bmatrix}$ $R_2 \longrightarrow R_2 - R_1$

(14) $2x+3y+5z=9$

$7x+3y-2z=5$

$2x+3y+\lambda z=\mu$

$$A=\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 5 \\ 2 & 3 & \lambda & \mu \end{array}\right] \quad R_1 \longrightarrow R_1 \times 1/2$$

$$=\left[\begin{array}{ccc|c} 1 & 3/2 & 5/2 & 9/2 \\ 7 & 3 & -2 & 5 \\ 2 & 3 & \lambda & \mu \end{array}\right] \quad \begin{array}{l} R_2 \longrightarrow R_2-7R_1 \\ R_3 \longrightarrow R_3-2R_1 \end{array}$$

$$=\left[\begin{array}{ccc|c} 1 & 3/2 & 5/2 & 9/2 \\ 0 & -15/2 & -39/2 & -53/2 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$$

No solⁿ

$\lambda=5$

$\mu\neq 9$

Unique Solution

$\lambda\neq 5$

$\mu\neq 9$.

Infinite solⁿ

$\lambda=5$

$\mu=9$

Differential Eq^n unit-3

Def^n:- A diff. Eq^n involving dependent variable, independent variable and derivatives of dependent variable w.r.t independent variable is called differential Eq^n.

eg.- $x^2 + \frac{dy}{dx} + 3y = 0$

$$\left(\frac{dy}{dx}\right)^2 = 3x$$

* Order and degree of D.E

Order — The order of D.E is the highest order of derivatives Present in a D.E.

e.g.- $\left(\frac{dy}{dx}\right)^2 + 3\left(\frac{d^2y}{dx^2}\right) + 5y = 0$ — order — 2

$$\frac{dy}{dx} = \frac{3x+4}{\frac{dy}{dx}}$$

$$\Rightarrow \frac{dy}{dx} \cdot \frac{dy}{dx} = 3x+4$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 3x+4$$ — order - 1.

Degree of D.E :-

The degree of D.E is the highest power of highest order derivatives.

e.g. - $\left(\frac{dy}{dx}\right)^2 + 3\left(\frac{d^2y}{dx^2}\right) + 5y = 0$ — order = 2 — degree = 1.

$\frac{d^2y}{dx^2} = K\left[1+\left(\frac{dy}{dx}\right)^2\right]$ — order = 2 — degree = 1

Imp. Homogenious D.E

A defferential equation is of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, where $f(x,y)$ and $g(x,y)$ are homogenious of same degree in x and y is called a homogeneous differential equation.

e.g. — $\frac{dy}{dx} = \frac{x^2-y^2}{xy}$

General solution of Homogenious and Non Homogenious Eqn :-

$Z = C.F + P.I$

complimentary factor (C.F)

Case - 1

If the Root of the equation is Real and Non-repeated.

C.F or $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$

Here, $c_1, c_2 \ldots c_n$ are constant.

$m_1, m_2 \ldots m_n$ are Real roots.

e.g :- Suppose the Root's are 1,2 & 3,

y_c or complimentary factor $= c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

Case - 2

If the Root of the equation is Real and Repeated.

let, $m_1 = m_2 = m_3$ are three roots.

$y_c = (c_1 + c_2 x + c_3 x^2) e^{m_1 x}$

e.g. Suppose the root of the equation is 2, 2, 2.

$$y_c = (c_1 + c_2 x + c_3 x^2) e^{2x}$$

$$y_c = (c_1 + c_2 x + c_2 x^2) e^{2x} + c_3 e^{3x}$$

Case - 3

If the Root of the equation is complex and non repeated.

Let, $m_1 = \alpha + i\beta$, $m_2 = \alpha + i\beta$.

C.F or $y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

e.g. - Suppose $m_1 = 2+3i$, $m_2 = 2-3i$,

$$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

Case - 4

If the Root of the equation is complex and repeated.

$m_1 = \alpha \pm i\beta$, $m_2 = \alpha \pm i\beta = a \pm ib$

C.F or $y_c = e^{\alpha x}\{(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\}$

e.g:— $m_1 = 1 \pm 2i$, $m_2 = 1 \pm 2i$

$\alpha = 1$, $\beta = 2$.

$$y_c = e^{x}\{(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x\}$$

Ex - 1

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

$D^2y - 5Dy + 6y = 0$

$y(D^2 - 5D + 6) = 0$

$\Rightarrow D^2 - 5D + 6 = 0$

$\Rightarrow D^2 - 3D - 2D + 6 = 0$

$\Rightarrow D(D-3) - 2(D-3) = 0$

$\Rightarrow (D-3)(D-2) = 0$

$D = 3 \quad D = 2$

$$y_c = c_1 e^{3x} + c_2 e^{2x}$$

Solve $\frac{d^2x}{dy^2} - 8\frac{dx}{dy} + 16x = 0$

$\Rightarrow D^2x - 8Dx + 16x = 0$

$\Rightarrow x(D^2 - 8D + 16) = 0$

$\Rightarrow D^2 - 8D + 16 = 0$

$\Rightarrow D^2 - 4D - 4D + 16 = 0$

$\Rightarrow D(D-4) - 4(D-4) = 0$

$\Rightarrow (D-4)(D-4) = 0$

$D = 4 \quad D = 4$

$x_c = (C_1 + C_2 x)e^{4x}$

① $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

② $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

③ $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

④ $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

⑤ $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Ans

① $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0.$

$\Rightarrow D^2y + 7Dy + 12y = 0$

$\Rightarrow y(D^2 + 7D + 12) = 0$

$\Rightarrow D^2 + 7D + 12 = 0.$ $\qquad y_c = (C_1 e^{-4x}) + C_2 e^{-3x}$

$\Rightarrow D^2 + 4D + 3D + 12 = 0$

$\Rightarrow D(D+4) + 3(D+4) = 0$

$D + 4 = 0 \quad D = -4 \quad D + 3 = 0 \quad D = -3$

(2) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$

$\Rightarrow D^2y+4Dy+13y=0$

$\Rightarrow y(D^2+4D+13)=0$

$\Rightarrow D^2+4D+13=0$

$D=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{-4\pm\sqrt{4^2-4\cdot1\cdot13}}{2\cdot1}$

$=\frac{-4\pm\sqrt{16-52}}{2}$

$=\frac{-4\pm\sqrt{-36}}{2}$

$\frac{-4+6i}{2},\frac{-4-6i}{2}$

$=\frac{2(-2+3i)}{2},\frac{2(-2-3i)}{2}$

$y_c=e^{-2x}(c_1\cos 3x+c_2\sin 3x)$

(4) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$

$\Rightarrow D^2y-2Dy+y=0$

$\Rightarrow y(D^2-2D+1)=0$

$\Rightarrow D^2-2D+1=0$

$\Rightarrow D^2-1D-1D+1=0$

$\Rightarrow D(D-1)-1(D-1)=0$

$\Rightarrow (D-1)(D-1)=0$

$D-1=0 \quad D-1=0$

$D=1 \quad D=1$

$y_c=(c_1+c_2x)e^x$

$y_c=(c_1e^{2x}+c_2e^{3x}+c_3e^x)$

(3) $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$

$\Rightarrow D^2y-3Dy+2y=0$

$\Rightarrow y(D^2-3D+2)=0$

$\Rightarrow D^2-3D+2=0$

$\Rightarrow D^2-2D-D+2=0$

$\Rightarrow D(D-2)-1(D-2)=0$

$\Rightarrow (D-2)(D-1)=0$

$D-2=0 \quad D-1=0$

$D=2 \quad D=1$

$y_c=(c_1e^{2x}+c_2e^x)$

(5) $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0$

$\Rightarrow D^3y-6D^2y+11Dy-6y=0$

$\Rightarrow y(D^3-6D^2+11D-6)=0$

$\Rightarrow D^3-6D^2+11D-6=0$

$\Rightarrow D^3-2D^2-4D^2+8D+3D-6=0$

$\Rightarrow D^2(D-2)-4D(D-2)+3(D-2)=0$

$\Rightarrow (D-2)(D^2-4D+3)=0$

$\therefore D-2=0 \Rightarrow \boxed{D=2}$

$D^2-4D+3=0$

$D^2-3D-D+3=0$

$\Rightarrow D(D-3)-1(D-3)=0$

$\Rightarrow (D-3)(D-1)=0$

$\Rightarrow D=3,1$

* $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

$D^2y + Dy + 1 = 0 \quad D^2 + D + 1 = 0$

$y(D^2 + D + 1) = 0$

$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1}$

$= \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{-3}}{2}$

$\frac{-1+\sqrt{3}i}{2} \quad \frac{-1-\sqrt{3}i}{2}$

$= \left(\frac{-1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)i$

$a + ib$

$y_c = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$

* $D^2y + 4Dy + 3y = 0$

$\Rightarrow y(D^2 + 4D + 3) = 0$

$\Rightarrow D^2 + 4D + 3 = 0$

$\Rightarrow D^2 + 3D + D + 3 = 0$

$\Rightarrow D(D+3) + 1(D+3) = 0$

$\Rightarrow (D+3)(D+1) = 0$

$D = -3 \quad D = -1$

$y_c = c_1 e^{-3x} + c_2 e^{-x}$

* $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

$\Rightarrow D^3y + 2D^2y + Dy = 0$

$\Rightarrow y(D^3 + 2D^2 + D) = 0$

$\Rightarrow D^3 + 2D^2 + D = 0$

$\Rightarrow D(D^2 + 2D + 1) = 0$

$\Rightarrow D(D^2 + 2D + 1) = 0$

$\Rightarrow (D)^2 + 2.D.1 + (1)^2 = 0$

$\Rightarrow D \cdot (D+1)^2 = 0$

$\Rightarrow D = 0 \quad (D+1) = 0 \quad (D+1) = 0$

$D + 1 = 0 \quad D + 1 = 0$

$D = -1 \quad D = -1$

$D = 0, -1, -1.$

$y_c = c_1 e^{0.x} + (c_2 + c_3 x)e^{-x}$

$= c_1 e^0 + (c_2 + c_3 x)e^{-x}$

$= c_1 + (c_2 + c_3 x)e^{-x}$

Particular Integral

$G.S = C.F + P.I$

Note $D = \frac{d}{dx}$ (derivative)

e.g.. $D(\sin x) = \cos x$ | $D^2(\cos x) = -\cos x$

$D(e^x) = e^x$

$$\boxed{\frac{1}{D} = \int f(x)\,dx \text{ (Integration)}}$$

e.g.. $\frac{1}{D}(\sin x) = -\cos x$

$$\frac{1}{D^2}(e^{2x}) = \int\int e^{2x}\,dx$$

$$= \int e^{2x}\,dx = \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{1}{2} \times \frac{e^{2x}}{2}$$

$$= \frac{e^{2x}}{4}$$

Case-I (Exponential Function)

Suppose $f(x) = e^{ax}$

$$P.I = \left[\frac{e^{ax}}{f(D)}\right] = \left[\frac{e^{ax}}{f(a)}\right]$$

If $f(a) = 0$

$$P.I = \frac{x \cdot e^{ax}}{f'(a)}$$

and so on. . . . .

e.g. Find P.I of $(D^2 + 3D + 2) = e^x$

Sol^n $P.I = \left(\frac{e^x}{D^2 + 3D + 2}\right)$

$$= \frac{e^x}{1+3+2} = \frac{e^x}{6}$$

e.g. – find P.I of $(D^2+16) = e^{-4x}$

$$P.I = \frac{e^{-4x}}{D^2+16}$$

$$= \frac{e^{-4x}}{(-4)^2+16}$$

$$= \frac{-e^{4x}}{16+16}$$

$$= \frac{-e^{4x}}{32}$$

e.g. :- $(D^2-4)y = e^{2x}$

$\Rightarrow (D^2-4)y = 0$

$\Rightarrow D^2-4 = 0$

$\Rightarrow D^2-2^2 = 0$

$\Rightarrow (D-2)(D+2) = 0$

$\Rightarrow D-2=0 \quad D+2=0$

$D = 2 \quad D = -2$

$$y_c = (c_1 e^{2x} + c_2 e^{-2x})$$

$$P.I = \frac{e^{2x}}{D^2-4}$$

$$= \frac{x \cdot e^{2x}}{2D}$$

$$= \frac{x \cdot e^{2x}}{4}$$

$\therefore$ General solution $= y_c + y_p$

$$= (c_1 e^{2x} + c_2 e^{-2x}) + \frac{x\, e^{2x}}{4}$$

e.g. – Solve, $(D-2)^2 = e^{2x}$

C.F :- $(D-2)^2 = 0$

$(D-2)(D-2) = 0$

$\Rightarrow D = 2 \quad D = 2$

$$y_c = (c_1 + c_2 x)\, e^{2x}$$

$$P.I = \frac{e^{2x}}{(D-2)^2}$$

$$= \frac{x\, e^{2x}}{2(D-2)}$$

$$= \frac{x^2\, e^{2x}}{2(1)}$$

$$= \frac{x^2\, e^{2x}}{2}$$

$\therefore$ General solution $= y_c + y_p$

$$= (c_1 + c_2 x)\, e^{2x} + \frac{x^2 e^{2x}}{2}$$

* Solve $(D^2-4)y = e^{2x} + e^{-4x}$

C.F $= (D^2-4)y = 0$

$\Rightarrow D^2-4 = 0$

$\Rightarrow D^2-2^2 = 0$

$\Rightarrow (D+2)(D-2) = 0$

$\Rightarrow D = -2 \quad D-2 = 0$

$D = 2$

$$y_c = (c_1 e^{2x} + c_2 e^{-2x})$$

$$P.I = y_p = \frac{e^{2x} + e^{-4x}}{D^2-4}$$

$$= \left(\frac{e^{2x}}{D^2-4}\right) + \left(\frac{e^{-4x}}{D^2-4}\right)$$

$$= \frac{x\, e^{2x}}{2D} + \frac{e^{-4x}}{4^2-4}$$

$$= \frac{x\, e^{2x}}{2\times 2} + \frac{e^{-4x}}{16-4}$$

$$= \frac{x\, e^{2x}}{4} + \frac{e^{-4x}}{12}$$

General Solution $= \left(c_1 e^{2x} + c_2 e^{-2x}\right) + \frac{x\, e^{2x}}{4} + \frac{e^{-4x}}{12}$

* Solve - $D^2y + 4Dy + 13y = 2xe^{-2x}$

$C.F = D^2y + 4Dy + 13y = 0$

$\Rightarrow y\,(D^2 + 4D + 13) = 0$

$\Rightarrow D^2 + 4D + 13 = 0$

$\Rightarrow a = 1,\ b = 4,\ c = 13$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4.1.13}}{2.1}$$

$$= \frac{-4 \pm \sqrt{16-52}}{2}$$

$$= \frac{-4 \pm \sqrt{-36}}{2}$$

$$= \frac{-4 \pm 6i}{2}$$

$= \frac{-4+6i}{2} = \frac{2(-2+3i)}{2}$, $= \frac{-4-6i}{2} = \frac{2(-2-3i)}{2}$

$y_c = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$

$$P.I = \frac{2e^{-x}}{D^2+4D+13}$$

$$= \frac{2e^{-x}}{(-1)^2+4(-1)+13}$$

$$= \frac{2e^{-x}}{1-4+13}$$

$$= \frac{2e^{-x}}{-3+13}$$

$$= \frac{\cancel{2}e^{-x}}{\cancel{10}\,5} = \frac{e^{-x}}{5}$$

$\therefore$ General solution

$$= e^{-2x}(c_1\cos 3x + c_2\sin 3x) + \frac{e^{-x}}{5}$$

~~HW~~

1 - $(D^3+3D^2+3D+1)y = e^{-x}$

2 - Solve $\frac{d^2y}{dx^2} + \frac{3dy}{dx} - 10y = 6e^{4x}$

3 - Solve $y'' + 5y' + 6y = e^{-2x}$

① $(D^3+3D^2+3D+1)y = e^{-x}$

$C.F = D^3+3D^2+3D+1 = 0$

$\cancel{\Rightarrow D^3+2D^2+D^2+2D+D+1=0}$

$\cancel{\Rightarrow D^2(D+2)+D(D+2)}$

$\Rightarrow (D+1)^3 = 0$

$\Rightarrow (D+1)(D+1)(D+1) = 0$

$D=-1 \quad D=-1 \quad D=-1$

$$\boxed{y_c = (c_1+c_2x+c_3x^2)e^{-x}}$$

$(D+1)^3$

$= D^3+3D^2+3D+1$

$\therefore$ General equation =

$$(c_1+c_2x+c_3x^2)e^{-x} + \frac{x^3e^{-x}}{6}$$

$$P.I = \frac{e^{-x}}{(D+1)^3}$$

$$= \frac{x\cdot e^{-x}}{3(D+1)^2} = \frac{x^2\cdot e^{-x}}{3\cdot 2(D+1)} = \frac{x^3e^{-x}}{6(1+0)} = \frac{x^3e^{-x}}{6}$$

(2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 6e^{4x}$

C.F $\Rightarrow D^2y + 3Dy - 10y = 0$

$\Rightarrow y(D^2+3D-10) = 0$

$\Rightarrow D^2+3D-10=0$

$\Rightarrow D^2+5D-2D-10=0$

$\Rightarrow D(D+5)-2(D+5)=0$

$\Rightarrow (D+5)(D-2)=0$

$\Rightarrow D+5=0 \quad , \quad D-2=0$

$D=-5 \qquad D=2$

$y_c = (c_1e^{-5x} + c_2e^{2x})$

$P.I = \frac{6e^{4x}}{D^2+3D-10}$

$= \frac{6e^{4x}}{(4)^2+3\cdot4-10}$

$= \frac{6e^{4x}}{16+12-10}$

$= \frac{6e^{4x}}{28-10}$

$= \frac{\cancel{6}e^{4x}}{\cancel{18}\,3} = \frac{e^{4x}}{3}$

General Solution

$= c_1e^{-5x} + c_2e^{2x} + \frac{e^{4x}}{3}$

(3) $y'' + 5y' + 6y = e^{-2x}$

C.F $\Rightarrow \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}$

$\Rightarrow D^2y + 5Dy + 6y = e^{-2x}$

$\cancel{\Rightarrow D^2y + 5}$

$\Rightarrow y(D^2+5D+6)=0$

$\Rightarrow D^2+5D+6=0$

$\Rightarrow D^2+3D+2D+6=0$

$\Rightarrow D(D+3)+2(D+3)=0$

$\Rightarrow (D+3)(D+2)=0$

$D+3=0 \qquad D+2=0$

$D=-3 \qquad D=-2$

$y_c = (c_1e^{-3x} + c_2e^{-2x})$

$P.I = \frac{e^{-2x}}{D^2+5D+6}$

$= \frac{e^{-2x}}{\cancel{(-2)^2+5\cdot(-2)+6}}$

$= \frac{e^{-2x}}{\cancel{4-10+6}}$

$= \frac{x\cdot e^{-2x}}{2D+5}$

$= \frac{x\cdot e^{-2x}}{2(-2)+5}$

$= \frac{x\cdot e^{-2x}}{-4+5}$

$= \frac{x\cdot e^{-2x}}{1}$

$\therefore$ General solution

$= c_1e^{-3x} + c_2e^{-2x} + \frac{xe^{-2x}}{1}$

Case - 2 (Trigonometry fun)

$$D\sin(ax+b) = a\cdot\cos(ax+b)$$

$$D^2\sin(ax+b) = a\ \cos(ax+b)$$

$$= a\left[-\sin(ax+b)\cdot a\right]$$

$$= -a^2\sin(ax+b)$$

To find the P.I;

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$$

$$f(-a^2)=0$$

Then

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{x\cdot\sin(ax+b)}{f'(D^2)}$$

Ex-1 Find P.I $(D^2+D+1)y = \sin 2x$

P.I.

$$\Rightarrow \cancel{D^2+D+1}\ y_p = \frac{\sin 2x}{D^2+D+1}$$

$$\Rightarrow \frac{\sin 2x}{(-2^2)+D+1}$$

$$\Rightarrow \frac{\sin 2x}{-4+D+1}$$

$$\Rightarrow \frac{\sin 2x\ (D+3)}{(D-3)(D+3)}$$

$$\Rightarrow \frac{\sin 2x\ (D+3)}{(D)^2-(3)^2}$$

$$\Rightarrow \frac{\sin 2x(D+3)}{(-2^2)-9}$$

$$= \frac{\sin 2x(D+3)}{-4-9}$$

$$= \frac{D\sin 2x + 3\sin 2x}{-13}$$

$$= \frac{2\cos 2x + 3\sin 2x}{-13}$$

Ex-2
solution

Evaluate P.I. of $\frac{1}{D^2+9}(4\sin 3x)$

$\Rightarrow y_p = \frac{4\sin 3x}{D^2+9}$ (case failure)

$\Rightarrow \frac{\cancel{4}^{2} x\sin 3x}{\cancel{2}D}$

$\Rightarrow \frac{2x\sin 3x}{D}$

$\Rightarrow \frac{1}{D} 2x\sin 3x$

$\Rightarrow 2x \frac{1}{D}\sin 3x$

$\Rightarrow 2x\int \sin 3x\, dx$

$\Rightarrow 2x\left(\frac{-\cos 3x}{3}\right)$

$\Rightarrow -\frac{2x}{3}(\cos 3x)$

Q) Find the P.I. of $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = \sin 2x$

$\Rightarrow D^2y + 5Dy + 6y = \sin 2x$

$\Rightarrow y(D^2+5D+6) = \sin 2x$

$\Rightarrow y_p = \frac{\sin 2x}{D^2+5D+6}$

$\Rightarrow y_p = \frac{\sin 2x}{-2^2+5D+6}$

$\Rightarrow y_p = \frac{\sin 2x}{-4+5D+6}$

$\Rightarrow y_p = \frac{\sin 2x}{5D+2}$

$$y_p = \frac{\sin 2x \,(5D-2)}{(5D+2)(5D-2)}$$

$$y_p = \frac{5D\sin 2x - 2\sin 2x}{(5D)^2 - (2)^2}$$

$$= \frac{5D\sin 2x - 2\sin 2x}{25D^2 - 4}$$

$$= \frac{5D\sin 2x - 2\sin 2x}{25\times -2^2 - 4}$$

$$= \frac{5D\sin 2x - 2\sin 2x}{25\times -4 - 4}$$

$$= \frac{5D\sin 2x - 2\sin 2x}{-100 - 4}$$

$$= \frac{5D\sin 2x - 2\sin 2x}{-104}$$

$$= \frac{5\times 2\cos 2x - 2\sin 2x}{-104}$$

$$= \frac{10\cos 2x - 2\sin 2x}{-104}$$

$$= \frac{2(5\cos 2x - \sin 2x)}{-104}$$

$$= \frac{5\cos 2x - \sin 2x}{-52}$$

H.W.

* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3\sin x$
* $(D^2+4)y = \sin 3x$
* $(D^4+D^3+D^2)y = \cos x$
* $(D^3-D)y = 4\cos x$

* $(D^2+4)y = \sin 3x$

$$y_p = \frac{\sin 3x}{D^2+4}$$

$$= \frac{\sin 3x}{(D)^2+4}$$

$$= \frac{\sin 3x}{-3^2+4}$$

$$= \frac{\sin 3x}{-9+4}$$

$$= \frac{\sin 3x}{-5}$$

* $(D^3-D)y = 4\cos x$

$$y_p = \frac{4\cos x}{D^3-D}$$

$$= \frac{4\cos x}{(D^2 \cdot D)-D}$$

$$= \frac{4\cos x}{(-1^2 \cdot D)-D}$$

$$= \frac{4\cos x}{-D-D}$$

$$= \frac{\cancel{2}4\cos x}{-\cancel{2}D}$$

$$= \frac{2\cos x}{-D}$$

$$= \frac{1}{-D} 2\cos x$$

$$= -2\frac{1}{D}\cos x$$

$$\cancel{= 2D\cos x}$$

$$\cancel{= 2(-\sin x) = -2\sin x}$$

$$= -2\sin x$$

$$= -2\sin x$$

* $(D^4+D^3+D^2)y = \cos x$

$$y_p = \frac{\cos x}{D^4+D^3+D^2}$$

$$= \frac{\cos x}{D^2 \cdot D^2 + D^2 \cdot D + D^2}$$

$$\cancel{= \frac{\cos x}{-1 + D + 1}}$$

$$\cancel{= \frac{\cos x}{2+D}}$$

$$= \frac{\cos x}{-1^2 \cdot -1^2 + -1^2 \cdot D + -1^2}$$

$$= \frac{\cos x}{(-1)(-1)+(-1)\cdot D + (-1)}$$

$$= \frac{\cos x}{1-D-1}$$

$$= \frac{\cos x}{-D}$$

$$= \frac{\cos x}{-D}$$

$$= \frac{1}{-D}\cos x$$

$$= -\frac{1}{D}\cos x$$

$$= \cancel{-\sin x}$$

$$= -(\sin x)$$

$$= -\sin x$$

* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3\sin x$

$\Rightarrow D^2y - 5Dy + 6y = 3\sin x$

$\Rightarrow y(D^2 - 5D + 6) = 3\sin x$

$$y_p = \frac{3\sin x}{D^2 - 5D + 6}$$

$$= \frac{3\sin x}{-1^2 - 5D + 6}$$

$$= \frac{3\sin x}{-1 - 5D + 6}$$

$$= \frac{3\sin x}{5 - 5D}$$

$$= \frac{3\sin x(-5D-5)}{(-5D+5)(-5D-5)}$$

$$= \cancel{\frac{-15D\sin x - 15\sin x}{}}$$

$$= \frac{3\sin x(-5D-5)}{(-5D)^2 - (5)^2}$$

$$= \frac{3\sin x(-5D-5)}{25D^2 - 25}$$

$$= \frac{3\sin x(-5D-5)}{25(-1^2) - 25}$$

$$= \frac{-15D\sin x - 15\sin x}{-25-25}$$

$$= \frac{-15\cos x - 15\sin x}{-50}$$

$$= \frac{-15(\cos x + \sin x)}{-50} = \frac{3(\cos x + \sin x)}{10}$$

~~Solve~~

* Solve $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=e^{-2x}+\sin 2x$

C.F $\Rightarrow D^2y+5Dy+6y=0$

$\Rightarrow y(D^2+5D+6)=0$

$\Rightarrow D^2+5D+6=0$

$\Rightarrow D^2+2D+3D+6=0$

$\Rightarrow D(D+2)+3(D+2)=0$

$\Rightarrow (D+2)(D+3)=0$

$D+2=0 \quad D+3=0$

$D=-2 \quad D=-3$

$y_c=(c_1e^{-2x}+c_2e^{-3x})$

$$P.I=\frac{e^{-2x}+\sin 2x}{D^2+5D+6}$$

$$=\frac{e^{-2x}}{D^2+5D+6}+\frac{\sin 2x}{D^2+5D+6}$$

$$=\frac{x\cdot e^{-2x}}{2D+5}+\frac{\sin 2x}{(-2^2)+5D+6}$$

$$=\frac{x\cdot e^{-2x}}{2(-2)+5}+\frac{\sin 2x}{-4+5D+6}$$

$$=\frac{x\cdot e^{-2x}}{-4+5}+\frac{\sin 2x}{5D+2}$$

$$=\frac{x\cdot e^{-2x}}{1}+\frac{\sin 2x\,(5D-2)}{(5D+2)(5D-2)}$$

$$=xe^{-2x}+\frac{\sin 2x\,(5D-2)}{(5D)^2-(2)^2}$$

$$= xe^{-2x} + \frac{5D\sin 2x - 2\sin 2x}{25D^2 - 4}$$

$$= xe^{-2x} + \frac{5\times 2\cos 2x - 2\sin 2x}{25(-2^2) - 4}$$

$$= xe^{-2x} + \frac{10\cos 2x - 2\sin 2x}{25(-4) - 4}$$

$$= xe^{-2x} + \frac{10\cos 2x - 2\sin 2x}{-100 - 4}$$

$$= xe^{-2x} + \frac{10\cos 2x - 2\sin 2x}{-104}$$

$$\therefore y = y_c + y_p$$

$$= (c_1 e^{-2x} + c_2 e^{-3x}) + xe^{-2x} + \frac{10\cos 2x - 2\sin 2x}{-104}$$

\# Solve $\frac{d^2y}{dx^2} + 4y = e^x + \sin 3x$

C.F

$\Rightarrow D^2y + 4y = 0$

$\Rightarrow y(D^2 + 4) = 0$

$\Rightarrow D^2 + 4 = 0$

$\cancel{\Rightarrow (D)^2 + (2)^2 = 0}$

$\Rightarrow D^2 = -4$

$\Rightarrow D = \sqrt{-4}$

$= \sqrt{4}i$

$= \pm 2i$

$= 2i, = -2i$

$y_c \Rightarrow e^0(c_1 \cos 2x + c_2 \sin 2x)$

$$y_p = \frac{e^x + \sin 3x}{D^2+4}$$

$$= \frac{e^x}{D^2+4} + \frac{\sin 3x}{D^2+4}$$

$$= \frac{e^x}{(1)^2+4} + \frac{\sin 3x}{(-3^2)+4}$$

$$= \frac{e^x}{1+4} + \frac{\sin 3x}{-9+4}$$

$$= \frac{e^x}{5} + \frac{\sin 3x}{-5}$$

$$= \frac{e^x}{5} - \frac{\sin 3x}{5}$$

$$y = y_c + y_p$$
$$= (c_1 \cos 2x + c_2 \sin 2x) + \left(\frac{e^x}{5} - \frac{\sin 3x}{5}\right)$$

Case-3

IMP. Binomial Expansion.

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4$ .... so on .....

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4$ .... so on .....

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4$ .... so on .....

$(1+x)^{-3} = 1 - 3x + 4x^2 - 5x^3 + 6x^4$ .... so on .....

Aljebraic function

* $F(D)y = x^n$

$\Rightarrow y_p = \frac{x^n}{F(D)}$

Taking constant common from denominator.

Ex - Solve. $(D^2+1)y = x$

C.F = $(D^2+1) = 0$

$D^2 = -1$

$D = \sqrt{-1}$

$= \pm 1i$

$= \pm i$

$y_c = (c_1 \cos x + c_2 \sin x)$

$P.I = \frac{x}{(D^2+1)}$

$= [1+D^2]^{-1} \cdot x$

$= [1 - D^2 + \cdots] x$

$= x$

Ex - Solve $\frac{d^2y}{dx^2} - 4y = x^2$

C.f :- $D^2y - 4y = 0$

$y(D^2-4) = 0$

$D^2 - 4 = 0$

$D^2 = 4$

$D = \pm\sqrt{4}$

$= \pm 2$

$2, -2,$

$y_c = (c_1 e^{2x} + c_2 e^{-2x})$

P.I = $D^2 - 4y = x^2$

$y(D^2-4) = x^2$

$y_p = \frac{x^2}{D^2-4}$

$= \frac{x^2}{-4\left(1 - \frac{D^2}{4}\right)}$

$= \frac{1}{-4}\left[1 - \frac{D^2}{4}\right]^{-1} \times x^2$

$= \frac{1}{-4}\left\{1 + \frac{D^2}{4} + \left(\frac{D^2}{4}\right)^2 + \cdots\right\} x^2$

$= \frac{1}{-4}\left(x^2 + \frac{D^2 \times x^2}{4} + \left(\frac{D^2}{4}\right)^2 + \cdots\right)$

$= \frac{1}{-4}\left[x^2 + 2/4 + 0 + \cdots\right)$

$= -1/4\left[x^2 + 2/4\right]$

$= \frac{x^2}{-4} - \frac{2}{16}$

Hence, General solution

$y = (c_1 e^{2x} + c_2 e^{-2x}) + \frac{x^2}{-4} - \frac{2}{16}$

* Solve $\frac{d^2y}{dx^2} + 4y = e^x + \sin 3x + x^2$

C.F: $D^2y + 4y = 0$

$\Rightarrow y(D^2+4) = 0$

$\Rightarrow D^2 + 4 = 0$

$\Rightarrow D^2 = -4$

$\Rightarrow D = \sqrt{-4}$

$= \pm 2i$

$y_c = (c_1 \cos 2x + c_2 \sin 2x)$

P.I $= D^2y + 4y = e^x + \sin 3x + x^2$

$\Rightarrow y(D^2+4) = e^x + \sin 3x + x^2$

$\Rightarrow y_p = \frac{e^x + \sin 3x + x^2}{D^2+4}$

$\Rightarrow \frac{e^x}{D^2+4} + \frac{\sin 3x}{D^2+4} + \frac{x^2}{D^2+4}$

$\Rightarrow \frac{e^x}{1^2+4} + \frac{\sin 3x}{-3^2+4} + \frac{x^2}{4(1+\frac{D^2}{4})}$

$\Rightarrow \frac{e^x}{5} + \frac{\sin 3x}{-9+4} + \frac{1}{4}(1+\frac{D^2}{4})^{-1} \cdot x^2$

$\Rightarrow \frac{e^x}{5} + \frac{\sin 3x}{-5} + \frac{1}{4}(x^2 + \frac{2}{4})$

$\Rightarrow \frac{e^x}{5} + \frac{\sin 3x}{-5} + \frac{x^2}{4} + \frac{2}{16}$

$\therefore$ General solution $=$

$(c_1 \cos 2x + c_2 \sin 2x) + \frac{e^x}{5} + \frac{\sin 3x}{-5} + \frac{x^2}{4} + \frac{2}{16}$

* Solve,

$(D^2 + 3D + 2)y = x^2$

C.F $= D^2 + 3D + 2 = 0$

$\Rightarrow D^2 + 2D + D + 2 = 0$

$\Rightarrow D(D+2) + 1(D+2) = 0$

$\Rightarrow (D+2)(D+1) = 0$

$D = -2 \quad D = -1$

$y_c = (c_1 e^{-2x} + c_2 e^{-x})$

P.I $= \frac{x^2}{D^2+3D+2}$

$= \frac{x^2}{2(1+\frac{D^2+3D}{2})}$

$= \frac{1}{2}\left(1 + \frac{D^2+3D}{2}\right)^{-1} \cdot x^2$

$= \frac{1}{2}\left[1 - \frac{D^2+3D}{2} + \left(\frac{D^2+3D}{2}\right)^2\right] \cdot x^2$

$= \frac{1}{2}\left(x^2 - \frac{2+3\cdot 2x}{2} + \frac{18}{4}\right)$

$= \frac{1}{2}\left(x^2 - \frac{2+6x}{2} + \frac{18}{4}\right)$

$$= \frac{1}{2}\left(\frac{4x^2-4-12x+18}{4}\right)$$

$$= \frac{4x^2-4-12x+18}{8}$$

$$= \frac{4x^2-12x+14}{8}$$

$$= \frac{2(2x^2-6x+7)}{8\,4}$$

$$= \frac{2x^2-6x+7}{4}$$

General solution

$$= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \frac{2x^2-6x+7}{4}$$

* $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1+x^2$

C.F

$\Rightarrow D^3y - D^2y - 6Dy = 0$

$\Rightarrow y(D^3 - D^2 - 6D) = 0$

$\Rightarrow D^3 - D^2 - 6D = 0$

$\Rightarrow D(D^2 - D - 6) = 0$

$\Rightarrow D^2 - D - 6 = 0$

$\boxed{D = 0}$

$\Rightarrow D^2 - 3D + 2D - 6 = 0$

$\Rightarrow D(D-3) + 2(D-3) = 0$

$\Rightarrow (D-3)(D+2) = 0$

$D = 3,\ D = -2$

$$y_c = \left(c_1 + c_2 e^{3x}\right) + c_3 e^{-2x}$$

$$P.I = \frac{1+x^2}{D^3 - D^2 - 6D}$$

$$\cancel{= \frac{1+x^2}{-6\left(1 - \frac{D^2+D}{6}\right)}}$$

$$= \frac{1+x^2}{-6D\left(1 + \frac{D^3 - D^2}{6D}\right)}$$

$$= \frac{1+x^2}{-6D\left[1 + \frac{D - D^2}{6}\right]}$$

$$= \frac{1}{-6D}\left[1 + \frac{D-D^2}{6}\right]^{-1} \cdot (1+x^2)$$

$$= \frac{1}{-6D}\left[1 - \left(\frac{D-D^2}{6}\right) + \left(\frac{D-D^2}{6}\right)^2\right](1+x^2)$$

$$= \frac{1}{-6D}\left[(1+x^2) - \left(\frac{D-D^2}{6}\right)(1+x^2) + \left(\frac{D^2-D^3}{36}\right)(1+x^2)\right]$$

$$= \frac{1}{-6D}\left[(1+x^2) - \frac{2x-2}{6} + \frac{2-0}{36}\right]$$

$$= \frac{1}{-6D}\left(\frac{(1+x^2)}{1} - \frac{2x-2}{6} + \frac{2}{36}\right]$$

$$= \frac{1}{-6D}\left(\frac{36+36x^2-12x+12+2}{36}\right)$$

$$= \frac{1}{-6D}\left[\frac{\cancel{2}(18+18x^2-6x+6+1)}{\cancel{36}\,18}\right]$$

$$\cancel{= \frac{1}{-6D}\left[1+x^2-\frac{1}{3}x\right.}$$

$$= \frac{1}{-6D}\left(\frac{18x^2-6x+25}{18}\right)$$

$$= \frac{1}{-108} \times \frac{1}{D}\left(18x^2-6x+25\right)$$

$$= \frac{1}{-108}\int(18x^2-6x+25)\,dx$$

$$= \frac{1}{-108}\left(\overset{6}{\cancel{18}}\left(\frac{x^3}{\cancel{3}}\right) - \cancel{6}\frac{3x^2}{\cancel{2}} + 25x\right)$$

$$= \frac{1}{-108}\left(6x^3-3x^2+25x\right)$$

General solution $= \left(C_1 + C_2 e^{3x}\right) + C_3 e^{-2x}\Big) + \frac{1}{-108}\left(6x^3-3x^2+25x\right)$

* Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x + x$

P.I.

$$D^3y + 2D^2y + Dy = e^{2x} + \sin 2x + x$$

$$\Rightarrow y(D^3 + 2D^2 + D) = e^{2x} + \sin 2x + x$$

$$\Rightarrow y_p = \frac{e^{2x} + \sin 2x + x}{D^3 + 2D^2 + D}$$

$$\Rightarrow y_p = \frac{e^{2x}}{D^3 + 2D^2 + D} + \frac{\sin 2x}{D^3 + 2D^2 + D} + \frac{x}{D^3 + 2D^2 + D}$$

$$= \frac{e^{2x}}{2^3 + 2\cdot 2^2 + 2} + \frac{\sin 2x}{D^2 \cdot D + 2D^2 + D} + \frac{x}{\cancel{D(D^2+2D+1)}\; D\left(1 + \frac{D^3 + 2D^2}{D}\right)}$$

$$= \frac{e^{2x}}{8 + 8 + 2} + \frac{\sin 2x}{-2^2 \cdot D + 2 \cdot -2^2 + D} + \frac{x}{D(1 + D^2 + 2D)}$$

$$= \frac{e^{2x}}{18} + \frac{\sin 2x}{-4D - 8 + D} + \frac{1}{D}(1 + D^2 + 2D)^{-1} x$$

$$= \frac{e^{2x}}{18} + \frac{\sin 2x\,(-3D + 8)}{(-3D - 8)(-3D + 8)} + \frac{1}{D}\left[x - (D^2 + 2D)x\right]$$

$$= \frac{e^{2x}}{18} + \frac{-3D\sin 2x + 8\sin 2x}{(-3D)^2 - (8)^2} + \frac{1}{D}\left[x - (0 + 2)\right]$$

$$= \frac{e^{2x}}{18} + \frac{-3D\sin 2x + 8\sin 2x}{9D^2 - 64} + \frac{1}{D}(x - 2)$$

$$= \frac{e^{2x}}{18} + \frac{-3D\sin 2x + 8\sin 2x}{9(-2^2) - 64} + \frac{x^2}{2} - 2x$$

$$= \frac{e^{2x}}{18} + \frac{-3D\sin 2x + 8\sin 2x}{9 \times -4 - 64} + \frac{x^2}{2} - 2x$$

$$= \frac{e^{2x}}{18} - \frac{6\cos 2x + 8\sin 2x}{-100} + \frac{x^2}{2} - 2x$$

Rule

If $f(D)y = e^{ax}V$

where, V is any function

$$y_p = \frac{e^{ax}.V}{f(D)}$$

$$= \frac{e^{ax}.V}{F(D+a)}$$

Eq.-1

solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$

$$D^2y + 5Dy + 6y = e^{-2x}\sin 2x$$

$$y(D^2+5D+6) = e^{-2x}\sin 2x$$

$$y_p = \frac{e^{-2x}\sin 2x}{D^2+5D+6}$$

$$= \left(\frac{e^{-2x}}{(D-2)^2+5(D-2)+6}\right)\sin 2x$$

$$= \left(\frac{e^{-2x}}{D^2-4D+4+5D-10+6}\right)\sin 2x$$

$$= \left(\frac{e^{-2x}}{D^2+D}\right)\sin 2x$$

$$= \left(\frac{\sin 2x}{-2^2+D}\right)e^{-2x}$$

$$= e^{-2x}\left(\frac{\sin 2x}{-4+D}\right)$$

$$= e^{-2x}\left(\frac{\sin 2x}{D-4}\right)$$

$$= e^{-2x}\left(\frac{\sin 2x(D+4)}{(D-4)(D+4)}\right)$$

$$= e^{-2x}\left(\frac{D\sin 2x+4\sin 2x}{D^2-4^2}\right)$$

$$= e^{-2x}\left(\frac{2\cos 2x+4\sin 2x}{(-2^2)-4^2}\right)$$

$$= e^{-2x}\left(\frac{2\cos 2x+4\sin 2x}{-4-16}\right)$$

$$= e^{-2x}\left(\frac{2\cos 2x+4\sin 2x}{-20}\right)$$

$$= e^{-2x}\left(\frac{\cancel{2}(\cos 2x+2\sin 2x)}{-\cancel{20}\,10}\right)$$

$$= e^{-2x}\left(\frac{\cos 2x+2\sin 2x}{-10}\right)$$

a) Solve, $D^2y+2y = e^x\cdot\cos 2x$

$$y(D^2+2) = e^x\cos 2x$$

$$y_p = \frac{e^x\cos 2x}{D^2+2}$$

$$= \left(\frac{e^x}{(D+1)^2+2}\right)\cos 2x$$

$$= \left(\frac{e^x}{D^2+2D+1+2}\right)\cos 2x$$

$$= \left(\frac{e^x}{D^2+2D+3}\right)\cos 2x$$

$$= \left(\frac{\cos 2x}{D^2+2D+3}\right) e^x$$

$$= \left(\frac{\cos 2x}{-2^2+2D+3}\right) e^x$$

$$= \left(\frac{\cos 2x}{-4+2D+3}\right) e^x$$

$$= \left(\frac{\cos 2x}{2D-1}\right) e^x$$

$$= \frac{\cos 2x\,(2D+1)}{(2D-1)(2D+1)}\, e^x$$

$$= \left(\frac{2D\cos 2x + \cos 2x}{(2D)^2-(1)^2}\right) e^x$$

$$= \left(\frac{2\times 2\sin 2x + \cos 2x}{4D^2-1}\right) e^x$$

$$= \left(\frac{-4\sin 2x + \cos 2x}{4\times -2^2-1}\right) e^x$$

$$= \left(\frac{-4\sin 2x + \cos 2x}{4\times -4-1}\right) e^x$$

$$= \frac{-4\sin 2x + \cos 2x}{-16-1}\, e^x$$

$$= \left(\frac{-4\sin 2x + \cos 2x}{-17}\right) e^x$$

(2) IMP

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^x$$

$$D^2y - 2Dy + y = x^2 e^x$$

$$y(D^2 - 2D + 1) = x^2 e^x$$

$$y_p = \frac{x^2 e^x}{D^2 - 2D + 1}$$

$$= \left(\frac{e^x}{(D+1)^2 - 2(D+1) + 1}\right) x^2$$

$$= \left(\frac{e^x}{D^2 + \cancel{2D} + \cancel{1} - \cancel{2D} - \cancel{2} + \cancel{1}}\right) x^2$$

$$= \left(\frac{e^x}{D^2}\right) x^2$$

$$= \left(\frac{x^2}{D^2}\right) e^x$$

$$= \left(\frac{1}{D^2} x^2\right) e^x$$

$$= \left(\frac{x^4}{3 \cdot 4}\right) e^x$$

$$= \left(\frac{x^4}{12}\right) e^x$$

$$\int \frac{x^3}{3} dx = \frac{1}{3}\int x^3 dx$$

$$= \frac{1}{3} \cdot \frac{x^4}{4}$$

$$= \frac{x^4}{12}$$

2). Find the c.f of $(D^2+3)y = e^{2x}$

$$P.I = \frac{e^{2x}}{D^3+3}$$

$$= \frac{e^{2x}}{2^3+3}$$

$$= \frac{e^{2x}}{8+3}$$

$$= \frac{e^{2x}}{11}$$

C.F = $D^3+3=0$

$D^3 = -3$

$D = \sqrt{-3}$

$= +\sqrt{3}i \quad -\sqrt{3}i$

~~$D^3+(\sqrt{3})^2=0$~~

$D^3 = -3$

$D^2 \cdot D = -3\times 1$

3). Solve $\frac{d^2y}{dx^2} + 2y = e^x \cos 2x$

$D^2y + 2y = e^x \cos 2x$

$y(D^2+2) = e^x \cos 2x$

C.F $= D^2+2=0$

$D^2 = -2$

$D = \sqrt{-2}$

$= \pm\sqrt{2}i$

$= +\sqrt{2}i \quad = -\sqrt{2}i$

$y_c = (c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$

$$P.I = \frac{e^x \cos 2x}{D^2+2}$$

$$= e^x \frac{1}{D^2+2} \cos 2x$$

$$= e^x \frac{1}{(D+1)^2+2} \cos 2x$$

$$= e^x \frac{1}{D^2+2D+1+2} \cos 2x$$

$$= \frac{\cos 2x}{D^2+2D+1} e^x$$

$$= \frac{\cos 2x}{-2^2 + 2D + 1} e^x$$

$$= \frac{\cos 2x}{-4 + 2D + 1} e^x$$

$$= \frac{\cos 2x}{2D - 3} e^x$$

$$= \frac{\cos 2x \, (2D + 3)}{(2D-3)(2D+3)} e^x$$

$$= \frac{2D\cos 2x + 3\cos 2x}{(2D)^2 - (3)^2} e^x$$

$$= \frac{-2 \cdot 2\sin 2x + 3\cos 2x}{4D^2 - 9} e^x$$

$$= \frac{-4\sin 2x + 3\cos 2x}{4(-2^2) - 9} e^x$$

$$= \frac{-4\sin 2x + 3\cos 2x}{4 \times -4 - 9} e^x$$

$$= \frac{-4\sin 2x + 3\cos 2x}{-16 - 9} e^x$$

$$= \frac{-4\sin 2x + 3\cos 2x}{-25} e^x$$

General solution $= (c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$

$$+ \frac{-4\sin 2x + 3\cos 2x}{-25} e^x$$

(Q4) Find the complimentary function if the roots of the auxillary eq^n s are - 0, -2, -2, -2

$$\cancel{\Rightarrow 2x^3 - 2x^2 - 2x = 0}$$

$$\cancel{= (C_1 + C_2}$$

$$= C_1 + (C_2 + C_3 x + C_4 x^2)\, e^{-2x}$$

Q5) Find the particular integral of the differential eq^n :-

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$$

$$\Rightarrow D^2 y - 2Dy + y = x e^x \sin x$$

$$\Rightarrow y(D^2 - 2D + 1) = x e^x \sin x$$

C.F = $D^2 - 2D + 1 = 0$

$$\Rightarrow D^2 - D - D + 1 = 0$$

$$\Rightarrow D(D-1) - 1(D-1) = 0$$

$$\Rightarrow (D-1)(D-1) = 0$$

$$\Rightarrow D - 1 = 0 \quad D - 1 = 0$$

$$D = 1 \quad D = 1$$

$$y_c = (c_1 + c_2 x)\, e^x$$

$$P.I = \frac{x e^x \sin x}{D^2 - 2D + 1}$$

Q6) P.I of $(D^2+16)y = e^{-4x}$

$$= \frac{e^{-4x}}{D^2+16}$$

$$= \cancel{\frac{x \cdot e^{-4x}}{2D}} = \frac{e^{-4x}}{(-4)^2+16}$$

$$= \frac{e^{-4x}}{16+16}$$

$$= \frac{e^{-4x}}{32}$$

Q.7) $(D^2+5D+6)y = e^{-2x}\sin 2x$

C.F $= D^2+5D+6=0$

$= D^2+3D+2D+6=0$

$= D(D+3)+2(D+3)=0$

$= (D+3)(D+2)=0$

$= (D+3)=0 \quad (D+2)=0$

$D=-2$

$D=-3$

$y_C = (c_1e^{-3x}+c_2e^{-2x})$

$$P.I = \frac{e^{-2x}\sin 2x}{D^2+5D+6}$$

$$= \frac{e^{-2x}}{D^2+5D+6}\sin 2x$$

$$= \frac{x\cdot e^{-2x}}{2D+5}\sin 2x$$

$$= \frac{x\cdot e^{-2x}(2D-5)}{(2D+5)(2D-5)}\sin 2x$$

$$= \frac{x\cdot e^{-2x}}{2x-2+5}\sin 2x$$

$$= \frac{x\cdot e^{-2x}}{-4+5}\sin 2x$$

$$= \frac{xe^{-2x}}{1}\sin 2x$$

e^{-2x}

$$P.I = \frac{e^{-2x}\sin 2x}{D^2+5D+6}$$

$$= \frac{e^{-2x}}{(D-2)^2+5(D-2)+6}\sin 2x$$

$$= \frac{e^{-2x}}{D^2-4D+4+5D-10+6}\sin 2x$$

$$= \frac{e^{-2x}}{D^2+D+10-10}\sin 2x$$

$$= \frac{e^{-2x}}{D^2+D}\sin 2x$$

$$= \frac{\sin 2x}{D^2+D}e^{-2x}$$

$$= \frac{\sin 2x}{-2^2+D}e^{-2x}$$

$$= \frac{\sin 2x}{D-4}e^{-2x}$$

$$= \frac{\sin 2x(D+4)}{(D-4)(D+4)}e^{-2x}$$

$$= \frac{D\sin 2x+4\sin 2x}{(D)^2-(4)^2}e^{-2x}$$

$$= \frac{-\cos 2x+4\sin 2x}{-2^2-16}e^{-2x}$$

$$= \frac{-\cos 2x+4\sin 2x}{-20}e^{-2x}$$

Q.8) $(D^2-2D-3)y = e^{3x}+\sin x$

C.F: $D^2-2D-3=0$

$\Rightarrow D^2-3D+D-3=0$

$\Rightarrow D(D-3)+1(D-3)=0$

$\Rightarrow (D-3)(D+1)=0$

$\Rightarrow D-3=0 \quad D+1=0$

$D=3 \quad D=-1$

$y_c = (c_1e^{3x}+c_2e^{-x})$

$$P.I = \frac{e^{3x}+\sin x}{D^2-2D-3}$$

$$= \frac{e^{3x}}{D^2-2D-3} + \frac{\sin x}{D^2-2D-3}$$

$$= \frac{x\cdot e^{3x}}{2D-2} \quad \cancel{= \frac{e^{3x}}{(3)^2-2\cdot 3-3}} + \frac{\sin x}{-1^2-2D-3}$$

$$= \frac{xe^{3x}}{2\cdot 3-2} \quad \cancel{= \frac{e^{3x}}{9-6-3}} + \frac{\sin x}{-1-2D-3}$$

$$= \frac{xe^{3x}}{6-2} + \frac{\sin x}{-2D-4}$$

$$= \frac{xe^{3x}}{4} + \frac{\sin x\,(-2D+4)}{(-2D-4)(-2D+4)}$$

$$= \frac{xe^{3x}}{4} + \frac{-2D\sin x+4\sin x}{(-2D)^2-(4)^2}$$

$$= \frac{xe^{3x}}{4} + \frac{-2\cos x+4\sin x}{4D^2-16}$$

$$= \frac{xe^{3x}}{4} + \frac{-2\cos x+4\sin x}{4\times -1^2-16}$$

$$= \frac{xe^{3x}}{4} + \frac{-2\cos x+4\sin x}{-4-16}$$

$$= \frac{xe^{3x}}{4} + \frac{-2\cos x+4\sin x}{-20}$$

$$= \frac{xe^{3x}}{4} - \frac{2\cos x-4\sin x}{20}$$

$\therefore$ General solution

$$= (c_1e^{3x}+c_2e^{-x}) + \frac{xe^{3x}}{4} - \frac{2\cos x-4\sin x}{20}.$$

Q9) C.F. of $(D^2+2D-15)y = \sin 3x$

$$(D^2+2D-15)y = \sin 3x$$

$$D^2+2D-15=0$$

$$D^2+5D-3D-15=0$$

$$D(D+5)-3(D+5)=0$$

$$(D+5)(D-3)=0$$

$$D+5=0 \quad D-3=0$$

$$D=-5 \quad D=3$$

C.F. =

$$y_c = (c_1 e^{-5x}) + c_2 e^{3x})$$

Q.10) Solve $(D^2+4D+3)y = e^{-x}\sin x$.

C.F. = $D^2+4D+3=0$

$$\Rightarrow D^2+3D+D+3=0$$

$$\Rightarrow D(D+3)+1(D+3)=0$$

$$\Rightarrow (D+3)(D+1)=0$$

$$D+3=0 \quad D+1=0$$

$$D=-3 \quad D=-1$$

$$y_c = (c_1 e^{-3x} + c_2 e^{-x})$$

$$P.I = \frac{e^{-x}\sin x}{D^2+4D+3}$$

$$= \frac{e^{-x}}{D^2+4D+3}\sin x$$

$$= \cancel{\frac{x e^{-x}}{2D+4}\sin x}$$

$$= \frac{e^{-x}}{(D-1)^2+4(D-1)+3}\sin x$$

$$= \frac{e^{-x}}{D^2-2D+1+4D-4+3}\sin x$$

$$= \frac{e^{-x}}{D^2+2D+\cancel{4-4}}\sin x$$

$$= \frac{\sin x}{2D+D^2}e^{-x}$$

$$= \frac{\sin x}{2D+-1^2}e^{-x}$$

$$= \frac{\sin x}{2D-1} e^{-x}$$

$$= \frac{\sin x\,(2D+1)}{(2D-1)(2D+1)} e^{-x}$$

$$= \frac{2D\sin x + \sin x}{(2D)^2-(1)^2} e^{-x}$$

$$= \frac{2\cos x + \sin x}{4D^2 - 1} e^{-x}$$

$$= \frac{2\cos x + \sin x}{4(-1) - 1} e^{-x}$$

$$= \frac{2\cos x + \sin x}{-4-1} e^{-x}$$

$$= \frac{2\cos x + \sin x}{-5} e^{-x}$$

General solution

$$= c_1 e^{-3x} + c_2 e^{-x} + \frac{2\cos x + \sin x}{-5} e^{-x}$$

Q) 11)

$$\frac{d^2y}{dx^3} - y = 0$$

$$D^2 y - y = 0$$

$$y(D^2 - 1) = 0$$

C.F $\quad D^2 - 1 = 0.$

$$D^2 = 1$$

$$D = \sqrt{1}$$

$$= \pm 1$$

$$= +1 \quad -1$$

$$y_c = (c_1 e^{x} + c_2 e^{-x})$$

Q 12) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$

$\Rightarrow D^2y - 5Dy + 6y = e^{2x}$

$\Rightarrow y(D^2 - 5D + 6) = e^{2x}$

$C.F = D^2 - 5D + 6 = 0$

$\Rightarrow D^2 - 3D - 2D + 6 = 0$

$\Rightarrow D(D-3) - 2(D-3) = 0$

$\Rightarrow (D-3)(D-2) = 0$

$D = 3 \quad D = 2$

$y_c = (c_1 e^{3x} + c_2 e^{2x})$

$P.I = \frac{e^{2x}}{D^2 - 5D + 6}$

$= \frac{x\, e^{2x}}{2D - 5}$

$= \frac{x\, e^{2x}}{2\cdot 2 - 5}$

$= \frac{x e^{2x}}{4 - 5}$

$= \frac{x\, e^{2x}}{-1}$

General solution $= (c_1 e^{3x} + c_2 e^{2x}) + \frac{x e^{2x}}{-1}$

Q 13) $(D^2+4)y = x^2 + \sin^2 x$

$C.F = D^2 + 4 = 0$

$\Rightarrow D^2 = -4$

$\Rightarrow D = \sqrt{-4}$

$= \pm 2i$

$= +2i, -2i$

$y_c = (c_1 \cos 2x + c_2 \sin 2x)$

General solution

$= (c_1 \cos 2x + c_2 \sin 2x) + \frac{x^2}{4} - \frac{1}{8} + \frac{\sin^2 x}{3}$

$P.I = \frac{x^2 + \sin^2 x}{D^2 + 4}$

$= \frac{x^2}{D^2+4} + \frac{\sin^2 x}{D^2+4}$

$= \frac{x^2}{4(1 + \frac{D^2}{4})} + \frac{\sin^2 x}{-1^2 + 4}$

$= \frac{1}{4}\frac{x^2}{(1 + \frac{D^2}{4})} + \frac{\sin^2 x}{-1+4}$

$= \frac{1}{4}\left(1 + \frac{D^2}{4}\right)^{-1} x^2 + \frac{\sin^2 x}{3}$

$= \frac{1}{4}\left(x^2 - \frac{D^2 x^2}{4} + \left(\frac{D^2}{4}\right)^2 x^2\right) + \frac{\sin^2 x}{3}$

$= \frac{1}{4}\left(x^2 - \frac{2}{4_2}\right) + \frac{\sin^2 x}{3}$

$= \frac{1}{4}\left(x^2 - \frac{1}{2}\right) + \frac{\sin^2 x}{3}$

$= \frac{x^2}{4} - \frac{1}{8} + \frac{\sin^2 x}{3}$

Q.14) $(D^2 - 2D + 2)y = \cos 3x$

$C.F = D^2 - 2D + 2 = 0$

$\Rightarrow \cancel{D^2 - D - D + 2 = 0}$

$\Rightarrow D(D-1) - 1$

$a = 1, b = -2, c = 2$

$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4.1.2}}{2.1}$

$= \frac{2 \pm \sqrt{4 - 8}}{2}$

$= \frac{2 \pm \sqrt{-4}}{2}$ $\quad \alpha + i\beta$

$= \frac{2 \pm 2i}{2} = \frac{2(1 \pm i)}{2}$

$= \frac{2 + 2i}{2}$ or $\frac{2 - 2i}{2}$.

$Y_c = e^{2x}(C_1 \cos 2x + C_2 \sin 2x)$

Q.15) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$

$\Rightarrow D^2y - 5Dy + 6y = e^{4x}$

$\Rightarrow y(D^2 - 5D + 6) = e^{4x}$

$C.F = D^2 - 5D + 6 = 0$

$\Rightarrow D^2 - 3D - 2D + 6 = 0$

$\Rightarrow D(D-3) - 2(D-3) = 0$

$\Rightarrow (D-3)(D-2) = 0$

$(D = 3)(D = 2)$

$Y_C = (C_1 e^{3x} + C_2 e^{2x})$

$P.I = \frac{e^{4x}}{D^2 - 5D + 6}$

$= \frac{e^{4x}}{(4)^2 - 5.4 + 6}$

$= \frac{e^{4x}}{16 - 20 + 6}$

$= \frac{e^{4x}}{22 - 20}$

$= \frac{e^{4x}}{2}$

General soln $= (C_1 e^{3x} + C_2 e^{2x}) + \frac{e^{4x}}{2}$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

Solve $y'' + 3y' + 2y = 4\cos^2 x$

C.F

$D^2y + 3Dy + 2y = 0$

$\Rightarrow y(D^2+3D+2) = 0$

$\Rightarrow D^2+3D+2 = 0$

$\Rightarrow D^2+D+2D+2 = 0$

$\Rightarrow D(D+1)+2(D+1) = 0$

$\Rightarrow (D+1)(D+2) = 0$

$D = -1 \quad D = -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

P.I.

$$y_p = \frac{4\cos^2 x}{D^2+3D+2}$$

$$= \frac{4\left[\frac{1+\cos 2x}{2}\right]}{D^2+3D+2}$$

$$= \frac{2[1+\cos 2x]}{D^2+3D+2}$$

$$= \frac{2 \cdot x^0}{D^2+3D+2} + \frac{2\cos 2x}{D^2+3D+2}$$

$$= \frac{\cancel{2}x^0}{\cancel{2}\left(1+\frac{D^2+3D}{2}\right)} + \frac{2\cos 2x}{-2^2+3D+2}$$

$$= \left[1+\frac{D^2+3D}{2}\right]^{-1} x^0 + \frac{2\cos 2x}{3D-2}$$

$$= \left[x^0 - \left(\frac{D^2-3D}{2}\right)x^0\right] + \frac{2\cos 2x\,(3D+2)}{(3D-2)(3D+2)}$$

$$= [x^0] + \frac{6D\cos 2x + 4\cos 2x}{(3D)^2 - (2)^2}$$

$$= [x^0] + \frac{6 \cdot 2\sin 2x + 4\cos 2x}{9D^2 - 4}$$

$$= [x^0] + \frac{-12\sin 2x + 4\cos 2x}{9(-2^2) - 4}$$

$$= 1 + \frac{-12\sin 2x + 4\cos 2x}{-36 - 4}$$

$$= 1 + \frac{-12\sin 2x + 4\cos 2x}{-40}$$

a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$

P.I $= D^2y - 2Dy + y = x e^x \sin x$

$\Rightarrow y(D^2 - 2D + 1) = x e^x \sin x$

$$y_p = \frac{x e^x \sin x}{D^2 - 2D + 1}$$

$$= x \sin x \frac{e^x}{(D+1)^2 - 2(D+1) + 1}$$

$$= x \sin x \frac{e^x}{D^2 + 2D + 1 - 2D - 2 + 1}$$

$$= x \sin x \frac{e^x}{D^2 + 2 - 2}$$

$$= x \sin x \frac{e^x}{D^2}$$

$= e^x \frac{x \sin x}{D^2}$

$= e^x \frac{1}{D^2} x \sin x$

$= \frac{e^x}{D} \cdot \frac{1}{D} x \sin x$

$= \frac{e^x}{D} \left\{ x \int \sin x \, dx - \int \left(\frac{d}{dx} x \times \int \sin x \, dx \right) dx \right\}$

$= \frac{e^x}{D} \left\{ -x \cos x - \int (1 \times -\cos x) \, dx \right\}$

$= \frac{e^x}{D} \left\{ -x \cos x + \sin x \right\}$

$= e^x \frac{1}{D} \sin x - x \cos x$

$= e^x \left\{ \int \sin x \, dx - \left[x \int \cos x \, dx - \int \left(\frac{d}{dx} x \times \int \cos x \, dx \right) dx \right] \right\}$

$= e^x \left\{ -\cos x - \left(x \sin x - \int (1 \times \sin x) \, dx \right) \right\}$

$= e^x \left\{ -\cos x - x \sin x + (-\cos x) \right\}$

$= e^x \left\{ -\cos x - x \sin x - \cos x \right\}$

$= e^x \left\{ -2 \cos x - x \sin x \right\}$

Partial differential equation

→ The partial diff is the relation between the independent and dependent variable.

→ If 'z' is a function of two variables x and y then its Partial derivatives are -

$$\frac{\partial z}{\partial x} = p \quad \frac{\partial^2 z}{\partial x^2} = r \quad \frac{\partial^2 z}{\partial x \partial y} = s$$

$$\frac{\partial z}{\partial y} = q \quad \frac{\partial^2 z}{\partial y^2} = t \quad \frac{\partial^2 z}{\partial y \partial x} = s$$

* Formation of a partial differential eqn (PDE) / Elemination of Arbitaray constant

Solve $z = ax + by + a^2 + b^2$

Soln $z = ax + by + a^2 + b^2$

$p = \frac{\partial x}{\partial z} = a$

$q = \frac{\partial z}{\partial y} = b$

$\therefore$ Hense $z = xp + qy + p^2 + q^2$

Ex-2

Solve $z = f(x^2 - y^2)$

$z = f(x^2 - y^2)$

$p = \frac{\partial z}{\partial x}$

$= \frac{\partial}{\partial x} f(x^2 - y^2)$

$= f'(x^2 - y^2) \frac{\partial}{\partial x} x^2 - y^2$

$= f'(x^2 - y^2) 2x$

$= 2x f'(x^2 - y^2)$ ——— (1)

$q = \frac{\partial z}{\partial y}$

$= \frac{\partial}{\partial y} f(x^2 - y^2)$

$= f'(x^2 - y^2) \frac{\partial}{\partial y} (x^2 - y^2)$

$= f'(x^2 - y^2) - 2y$

$= -2y f'(x^2 - y^2)$ ——— (2)

$\therefore$ Dividing eq^n (1) and (2)

$$\Rightarrow \frac{p}{q} = \frac{\cancel{2}x f'(\cancel{x^2-y^2})}{-\cancel{2}y f'(\cancel{x^2-y^2})}$$

$$\Rightarrow \frac{p}{q} = \frac{-x}{y}$$

$$\Rightarrow py = -qx$$

$$\Rightarrow py + qx = 0$$

Ex-3 Solve $z = f(xy/z)$

$$p = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(xy/z)$$

$$= f'(xy/z)\, \frac{\partial}{\partial x} \frac{xy}{z}$$

$$= f'(xy/z)\, \frac{y}{z} \frac{\partial}{\partial x} x$$

$$= f'(xy/z)\, y/z$$

$$= y/z\, f'(xy/z)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(xy/z)$$

$$= f'(xy/z)\, \frac{\partial}{\partial y} xy/z$$

$$= f'(xy/z)\, x/z\, \frac{\partial}{\partial y} y$$

$$= \frac{x}{z} f'(xy/z)$$

Dividing eq^n (1) and (2)

$$\Rightarrow \frac{p}{q} = \frac{y/z\, \cancel{f'(xy/z)}}{x/z\, \cancel{f'(xy/z)}}$$

$$\Rightarrow \frac{p}{q} = \frac{y/z}{x/z}$$

$$\Rightarrow p\, x/\cancel{z} = q\, y/\cancel{z}$$

$$\cancel{\Rightarrow p\, x/z - q\, y/z = 0}$$

$$\Rightarrow px - qy = 0$$

IMP

(2) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$$p = \frac{\partial}{\partial x} \frac{x^2}{2a^2} + \frac{y^2}{2b^2}$$

$$= \frac{\partial}{\partial x} \frac{x^2}{2a^2} + \frac{\partial}{\partial x} \frac{y^2}{2b^2}$$

$$= \frac{1}{2a^2} 2x + 0$$

$$= 2x \frac{1}{2a^2}$$

$$= \frac{x}{a^2}$$

$$\cancel{xp = \frac{x^2}{a^2}}$$

$$\cancel{yp = \frac{2y^2}{b^2}}$$

$$\cancel{2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}}$$

$\cancel{2 \quad 2}$

$$q = \frac{\partial}{\partial y} \frac{x^2}{2a^2} + \frac{y^2}{2b^2}$$

$$= \frac{\partial}{\partial y} \frac{x^2}{2a^2} + \frac{\partial}{\partial y} \frac{y^2}{2b^2}$$

$$= 0 + \frac{1}{2b^2} \frac{\partial}{\partial y} y^2$$

$$= \frac{1}{2b^2} 2y$$

$$= 2y \frac{1}{2b^2}$$

$$= \frac{2y}{b^2}$$

$$xp = \frac{x^2}{a^2} \qquad yp = \frac{y^2}{b^2}$$

$$\boxed{2z = xp + yq}$$

Linear Eqn of 1st order -

→ A linear differential Eqn of 1st order is known as Lagrange's linear Eqn is of the form

$$\boxed{Pp + Qq = R}$$

To solve the linear D.E

$Pp + Qq = R$ is $f(u,v) = 0$ or $\phi(u,v) = 0$.

$$\therefore \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

* Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

here $P = \sqrt{x}$

$Q = \sqrt{y}$

$R = \sqrt{z}$.

It's auxillary equation is :-

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

First comparing 1st and 2nd

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$$

Integrating both sides -

$$\Rightarrow \int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$$

$$\Rightarrow \int \frac{1}{x^{1/2}} dx = \int \frac{1}{y^{1/2}} dy$$

$$\Rightarrow \int x^{-1/2} dx = \int y^{-1/2} dy$$

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \cancel{dx} = \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \cancel{dy}$$

$$\Rightarrow \frac{x^{1/2}}{1/2} = \frac{y^{1/2}}{1/2}$$

$$\Rightarrow \cancel{2}x^{1/2} = \cancel{2}y^{1/2}$$

$$\Rightarrow \sqrt{x} - \sqrt{y} = 0 \quad \text{——— (1)}$$

Again comparing 2nd and 3rd -

$$\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

$$\Rightarrow \int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$$

$$\Rightarrow \int \frac{dy}{y^{1/2}} = \int \frac{dz}{z^{1/2}}$$

$$\Rightarrow \int y^{-1/2}\, dy = \int z^{-1/2}\, dz$$

$$\Rightarrow \frac{y^{-1/2+1}}{-1/2+1} = \frac{z^{-1/2+1}}{-\frac{1}{2}+1}$$

$$\Rightarrow \frac{y^{1/2}}{1/2} = \frac{z^{1/2}}{1/2}$$

$$\Rightarrow \cancel{2}y^{1/2} = \cancel{2}z^{1/2}$$

$$\Rightarrow \sqrt{y} - \sqrt{z} = 0 \quad \text{——— (2)}$$

$$\therefore \phi(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0.$$

* Solve $pyz + qzx = xy$.

$pyz + qzx = xy$.

Here, $P = yz$, $Q = zx$, $R = xy$.

Then the auxillary equation is,-

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}.$$

On comparing 1st and 2nd ratio-

$$\frac{dx}{yz} = \frac{dy}{zx}$$

$\Rightarrow dx \times \cancel{z}x = dy \cdot y\cancel{z}$

$\Rightarrow x\,dx = y\,dy$

$\Rightarrow \int x\,dx = \int y\,dy$

$\Rightarrow \frac{x^2}{\cancel{2}} = \frac{y^2}{\cancel{2}}$

$\Rightarrow x^2 - y^2 = 0$ ——— (1)

On comparing 2nd and 3rd ratio -

$$\frac{dy}{zx} = \frac{dz}{xy}$$

~~$\Rightarrow dy \times xy = dz \times zx$~~ $\Rightarrow \cancel{x}y \times dy = dz \times z\cancel{x}$

~~$\Rightarrow xy^2 = xz^2$~~ $\Rightarrow \int y\,dy = \int z\,dz$

~~$\Rightarrow y^2 = z^2$~~ $\Rightarrow \frac{y^2}{\cancel{2}} = \frac{z^2}{\cancel{2}}$

$\Rightarrow y^2 - z^2 = 0$ ——— (2).

$\therefore \phi(x^2 - y^2, y^2 - z^2) = 0$.

IMP

Ex-3 Solve $p \tan x + q \tan y = \tan z$

$P = \tan x$, $Q = \tan y$, $R = \tan z$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Comparing 1st and 2nd :-

$\Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y}$

$\Rightarrow \int \frac{dx}{\tan x} = \int \frac{dy}{\tan y}$

$\Rightarrow \int \cot x\,dx = \int \cot y\,dy$

$\Rightarrow \log(\sin x) = \log(\sin y)$

$\Rightarrow \log(\sin x) - \log(\sin y) = 0$

$\Rightarrow \log \dfrac{\sin x}{\sin y} = 0$ ——— (1).

comparing (2) and (3).

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$\Rightarrow \int \cot y\,dy = \int \cot z\,dz$

$\Rightarrow \ln(\sin y) = \ln(\sin z)$

$\Rightarrow \ln(\sin y) - \ln(\sin z) = 0$

$\Rightarrow \ln \dfrac{(\sin y)}{(\sin z)} = 0$ ——— (2).

$\therefore \phi\left(\ln \dfrac{\sin x}{\sin y}, \ln \dfrac{\sin y}{\sin z}\right)$.

Imp
Solve. $(y+z)p-(z+x)q=x-y$

$\Rightarrow (y+z)p-(z+x)q=x-y.$

$P=y+z \quad ,Q=-(z+x)\ ,\ R=x-y$

Then the Auxilliary eqn is

$$\frac{dx}{y+z}=\frac{dy}{-(z+x)}=\frac{dz}{x-y}.$$

$\therefore \cancel{\dfrac{dx-dy+dz}{y+z-z-x+x-y}}$ Here, $P'=1\ ,Q'=1,\ R'=1$

$PP'+QQ'+RR'=y+z+(-z-x)+x-y$

Then, $P'dx+Q'dy+R'dz=0$

$\Rightarrow dx+dy+dz=0$

$\Rightarrow \int dx+\int dy+\int dz=0$

$\Rightarrow x+y+z=c_1$ ——— (1)

$P''=x,\ Q''=y\ ,R''=-z$

$PP''+QQ''+RR''=(y+z)x-(z+x)y+(x-y)z$

$=\cancel{yx+zx-zy-xy-zx+yz}$

$=0$

$P''dx+Q''dy+R''dz=0$

$\Rightarrow \int x\,dx+\int y\,dy-\int z\,dz=0$

$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2}=C_2$ ——— (2)

$\Rightarrow \frac{x^2+y^2-z^2}{2}=C_2$

$\Rightarrow x^2+y^2-z^2=2C_2$ ——— (2)

$\Rightarrow \phi(x+y+z, x^2+y^2-z^2)$

IMP Solve. $x(y-z)p + y(z-x)q = z(x-y)$

$P = x(y-z)$, $Q = y(z-x)$, $R = (x-y)z$

$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{(x-y)z}$

Here, $P' = 1$, $Q' = 1$, $R' = 1$

$\Rightarrow PP' + QQ' + RR' = xy - xz + yz - xy + zx - yz$
$= 0$

Then $P'dx + Q'dy + R'dz = 0$

$\Rightarrow dx + dy + dz = 0$

$\Rightarrow \int dx + \int dy + \int dz = 0$

$\Rightarrow x+y+z = c_1$ ——— (1)

$P'' = \frac{1}{x}$, $Q'' = \frac{1}{x}$, $R'' = \frac{1}{x}$.

$PP'' + QQ'' + RR'' = \frac{x(y-z)}{x} + \frac{y(z-x)}{y} + \frac{z(x-y)}{z}$
$= y - z + z - x + x - y$
$= 0$

$P''dx + Q''dy + R''dz = 0$

$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{x}dy + \int \frac{1}{x}dz = 0$ $\quad \therefore \phi(x+y+z, xyz)$

$\Rightarrow \log x + \log y + \log z = c_2$ ——— (2)

~~$\Rightarrow \log x+y+z =$~~

$\Rightarrow \log xyz = \log c_2$ ——— (2)

$\Rightarrow xyz = c_2$

* Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$P = x^2(y-z)$ $\quad Q = y^2(z-x)$ $\quad R = z^2(x-y)$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

$P' = \frac{1}{x}, \; Q' = \frac{1}{y}, \; R' = \frac{1}{z}$

$\Rightarrow PP' + QQ' + RR' = 0$

$\Rightarrow x^2(y-z) \times \frac{1}{x} + y^2(z-x) \times \frac{1}{y} + z^2(x-y) \times \frac{1}{z} = 0$

$\Rightarrow x(y-z) + y(z-x) + z(x-y) = 0$

$\Rightarrow \cancel{xy} - \cancel{zx} + \cancel{yz} - \cancel{yx} + \cancel{zx} - \cancel{yz} = 0$

Then,

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = c_1$$

$$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = c_1$$

$$\Rightarrow \ln x + \ln y + \ln z = c_1$$

$$\Rightarrow \ln xyz = \ln c_1$$

$$\Rightarrow xyz = c_1 \text{ ——— } (1)$$

$P'' = \frac{1}{x^2}, \; Q'' = \frac{1}{y^2}, \; R'' = \frac{1}{z^2}$

$$\Rightarrow PP'' + QQ'' + RR'' = \frac{\cancel{x^2}(y-z)}{\cancel{x^2}} + \frac{\cancel{y^2}(z-x)}{\cancel{y^2}} + \frac{\cancel{z^2}(x-y)}{\cancel{z^2}}$$

$$= \cancel{y} - \cancel{z} + \cancel{z} - \cancel{x} + \cancel{x} - \cancel{y}$$

$$= 0$$

Then,

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = c_2$$

$$\Rightarrow \int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = c_2$$

$\Rightarrow \int x^{-2}dx + \int y^{-2}dy + \int z^{-2}dz = c_2$

$\Rightarrow \frac{x^{-2+1}}{-2+1} + \frac{y^{-2+1}}{-2+1} + \frac{z^{-2+1}}{-2+1} = c_2$

$\Rightarrow \frac{x^{-1}}{-1} + \frac{y^{-1}}{-1} + \frac{z^{-1}}{-1} = c_2$

$\Rightarrow -\frac{1}{x} + -\frac{1}{y} + -\frac{1}{z} = c_2$

$\Rightarrow -\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = c_2$ ——— (2)

$\therefore \phi\left(xyz, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$.

* Solve $x(y^2-z^2)p + y(z^2-x^2)q - z(x^2-y^2) = 0$

~~$P = y^2-z^2, Q = z^2-x^2, z = -(x^2-y^2)$~~

$\Rightarrow x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

$P' = \frac{1}{x}, \; Q' = \frac{1}{y}, \; R' = \frac{1}{z}$

$\Rightarrow PP' + QQ' + RR'$

$= \frac{x(y^2-z^2)}{x} + \frac{y(z^2-x^2)}{y} + \frac{z(x^2-y^2)}{z}$

~~$= xy^2 - xz^2 + yz^2 - yx^2 + zx^2 - zy^2$~~

$\Rightarrow y^2 - z^2 + z^2 - x^2 + x^2 - y^2$

$= 0$

Then, $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = c_1$

$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = c_1$

$\Rightarrow \ln x + \ln y + \ln z = \ln c_1$

$\Rightarrow \ln xyz = \ln c_1$

$\Rightarrow xyz = c_1$ — (1)

$P'' = x, \quad Q'' = y, \quad R'' = z$

$\Rightarrow x[x(y^2-z^2)] + y[y(z^2-x^2)] + z[z(x^2-y^2)]$

$= x(xy^2 - xz^2) + y(yz^2 - yx^2) + z(zx^2 - zy^2)$

$\Rightarrow \cancel{x^2y^2} - \cancel{x^2z^2} + \cancel{y^2z^2} - \cancel{y^2x^2} + \cancel{z^2x^2} - \cancel{z^2y^2}$

$= 0.$

Then, $x\,dx + y\,dy + z\,dz = C_2$

$\Rightarrow \int x\,dx + \int y\,dy + \int z\,dz = C_2$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2$

$\Rightarrow \frac{x^2+y^2+z^2}{2} = C_2$

$\Rightarrow x^2+y^2+z^2 = C_2$ ——— (2)

$\therefore \phi(xyz, x^2+y^2+z^2)$ ..

Laplace Transformation (unit-4)

If the laplace Transformation of $f(s)$ is $f(t)$ i.e

$$L\{f(t)\} = f(s)$$

$$\boxed{L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt}$$, where 's' is a variable.

Some fudamental formula—

(1) $L\{1\} = \frac{1}{s}$

(2) $L\{t^n\} = \frac{n!}{s^{n+1}}$

$L\{t\} = \frac{1}{s^2}$

$L\{t^2\} = \frac{2}{s^3}$

$L\{t^3\} = \frac{6}{s^4}$

(3) $L\{e^{at}\} = \frac{1}{s-a}$

Ex- $L\{e^{2t}\} = \frac{1}{s-2}$

(4) $L\{e^{-at}\} = \frac{1}{s+a}$

$L\{e^{-3t}\} = \frac{1}{s+3}$

(5) $L\{\sin(at+b)\}$

$= \frac{a}{s^2+a^2}$

Ex-1

$L\{\sin 2t\}$

$= \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$

(6) $L\{\cos(at+b)\} = \frac{s}{s^2+a^2}$

Ex

$L\{\cos(3t+1)\}$

$= \frac{s}{s^2+3^2} = \frac{s}{s^2+9}$

(7) $L\{\sinh at\}$

$= \frac{a}{s^2-a^2}$

(8) $L\{\cosh at\}$

$= \frac{s}{s^2+a^2}$

$$\left\{\begin{array}{l} \cosh at = \dfrac{e^{at}+e^{-at}}{2} \\ \sinh at = \dfrac{e^{at}-e^{-at}}{2} \end{array}\right\}$$

1) $L\{8\} = 8/s$

2) $L\{e^{4t}\} = \frac{1}{s-4}$

3) $L\{4e^{5t}\} = 4L\{e^{5t}\}$

$= 4 \times \frac{1}{s-5}$

$= \frac{4}{s-5}$

4) $L\{3\sinh 3t\}$

$= 3L\{\sinh 3t\}$

$= 3 \times \frac{3}{s^2-3^2}$

$= \frac{9}{s^2-9}$

5) $L\{2\cosh 2t\}$

$= 2L\{\cosh 2t\}$

$= 2 \times \frac{s}{s^2-2^2}$

$= \frac{2s}{s^2-2^2}$

Ex-1

Find the laplace transform of $1+2t^3-4e^{3t}+5e^{-t}$

$\Rightarrow L\{1+2t^3-4e^{3t}+5e^{-t}\}$

$\Rightarrow L\{1\}+L\{2t^3\}-L\{4e^{3t}\}+L5\{e^{-t}\}$

$\Rightarrow \frac{1}{s}+2L\{t^3\}-4L\{e^{3t}\}+5L\{e^{-t}\}$

$\Rightarrow \frac{1}{s}+2\times\frac{6}{s^4}-4\left(\frac{1}{s-3}\right)+5\left(\frac{1}{s+1}\right)$

$\Rightarrow \frac{1}{s}+\frac{12}{s^4}-\frac{4}{s-3}+\frac{5}{s+1}$

Ex-2

Find the laplace transform of $3\cosh 4t + 4\sin 3t$

$\Rightarrow L\{3\cosh 4t\}+L\{4\sin 3t\}$

$\Rightarrow 3L\{\cosh 4t\}+4L\{\sin 3t\}$

$\Rightarrow 3\times\frac{s}{s^2-4^2}+4\times\frac{3}{s^2+3^2}$

$\Rightarrow \frac{3s}{s^2-16}+\frac{12}{s^2+9}$

$$\Rightarrow \frac{3s(s^2+9)+12(s^2-16)}{(s^2-16)(s^2+9)}$$

$$\Rightarrow \frac{3s^3+27s+12s^2-192}{(s^2-16)(s^2+9)}$$

$$\Rightarrow \frac{3s^3+12s^2+27s-192}{(s^2-16)(s^2+9)}$$

Ex = Find $L\{t - \sinh 2t\}$

$$\Rightarrow L\{t\} - L\{\sinh 2t\}$$

$$\Rightarrow \frac{1}{s^2} - \frac{2}{s^2-2^2}$$

$$\Rightarrow \frac{s^2-2^2-2s^2}{s^2(s^2-2^2)}$$

$$\Rightarrow \frac{\cancel{+s^2}-4}{\cancel{s^4}-s^2 4} \Rightarrow \frac{-s^2-4}{s^2(s^2-4)}$$

$$\Rightarrow \frac{-4\cancel{-s^2}}{s^4-4s^2} \Rightarrow \frac{-(s^2+4)}{s^2(s^2-4)}$$

IMP

Ex = 4

find $L\{(\sin t - \cos t)^2\}$

$$= L\{(\sin t - \cos t)^2\}$$

$$= L\{\sin^2 t + \cos^2 t - 2\sin t \cdot \cos t\}$$

$$= L\{1 - \sin 2t\}$$

$$= \frac{1}{s} - \frac{2}{s^2+2^2}$$

$$= \frac{1}{s} - \frac{2}{s^2+4}$$

$$= \frac{s^2+4-2s}{s(s^2+4)}$$

$$= \frac{s^2-2s+4}{s(s^2+4)}$$

(1)

(a) $3e^{-2t} - 2e^{3t}$

(b) $2t^2 - 3t + 4$

(c) $\cos(at+b)$

(d) $t - \sinh 2t$

IMP (e) $\sin 2t \cdot \cos 3t$

(1) (a) $3e^{-2t} - 2e^{3t}$

$= 3\,L(e^{-2t}) - 2\,L(e^{3t})$

$= 3\left(\frac{1}{s+2}\right) - 2\left(\frac{1}{s-3}\right)$

$= \frac{3}{s+2} - \frac{2}{s-3}$

$= \frac{(s-3)3 - 2(s+2)}{(s+2)(s-3)}$

$= \frac{3s-9-2s-4}{(s+2)(s-3)}$

$= \frac{s-13}{(s+2)(s-3)}$

(b) $L\{2t^2 - 3t + 4\}$

$= 2L[t^2] - 3[t] + L(4)$

$= 2\frac{2}{s^3} - 3\frac{1}{s^2} + 4/s$

$= \frac{4}{s^3} - \frac{3}{s^2} + 4/s$

(c) $L\{\cos(at+b)\}$

$= \frac{s}{s^2+a^2}$

(d) $t - \sinh 2t$

$= \frac{1}{s^2} - \frac{2}{s^2-4}$

* $L\{\sin 2t \cdot \cos 3t)$

$= \frac{1}{2}[\sin(2t+3t) + \sin(2t-3t)]$

$= \frac{1}{2}[\sin 5t + \sin(-t)]$

$= \frac{1}{2}[\sin 5t - \sin t]$

$\therefore L\{\sin 2t \cdot \cos 3t\}$

$= \frac{1}{2} L\{\sin 5t - \sin t\}$

$= \frac{1}{2} L\{\sin 5t\} - L\{\sin t\}$

$= \frac{1}{2}\left\{\frac{5}{s^2+25} - \frac{1}{s^2+1}\right)$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

$\Rightarrow \left\{\frac{1}{2}[\sin(A+B) + \sin(A-B)] = \sin A \cdot \cos B\right.$

$\sin(-\theta) = -\sin\theta$.

* $L\{\cos at \cdot \cos bt\}$

$\Rightarrow \frac{1}{2}[\cos(A+B) + \cos(A-B)] = \cos A \cdot \cos B$

$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$
$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

$\cos(A+B) + \cos(A-B) = 2\cos A \cdot \cos B$.

$\Rightarrow \left\{\frac{1}{2}[\cos(A+B) + \cos(A-B)] = \cos A \cdot \cos B\right.$

$\Rightarrow \cos at \cdot \cos bt$

$\Rightarrow \frac{1}{2}[\cos(at+bt) + \cos(at-bt)]$

$\Rightarrow \frac{1}{2}[\cos(a+b)t + \cos(a-b)t]$

$\therefore L\{\cos at \cdot \cos bt\} = \frac{1}{2} L[\cos(a+b)t + \cos(a-b)t]$

$= \frac{1}{2}\{L[\cos(a+b)t] + L[\cos(a-b)t]\}$

$= \frac{1}{2}\left[\frac{s}{s^2+(a+b)^2} + \frac{s}{s^2+(a-b)^2}\right]$

* Find $L\{\cos^2 at\}$

$= L\{\cos^2 at\}$

$= L\left\{\frac{1+\cos 2at}{2}\right\}$

$= \frac{1}{2} L\{1+\cos 2at\}$

$= \frac{1}{2}\left[L\{1\} + L\{\cos 2at\}\right]$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4a^2}\right]$

$= \frac{1}{2}\left[\frac{s^2+4a^2+s^2}{s(s^2+4a^2)}\right]$

$= \frac{1}{2}\left[\frac{2s^2+4a^2}{s(s^2+4a^2)}\right]$

$= \frac{2s^2+4a^2}{2s(s^2+4a^2)} = \frac{\cancel{2}(s^2+2a^2)}{\cancel{2}s(s^2+4a^2)}$

$= \frac{s^2+2a^2}{s(s^2+4a^2)}$

$\cos 2\theta = 2\cos^2\theta - 1$ $\quad 1+\cos 2\theta = 2\cos^2\theta$

$\frac{1+\cos 2\theta}{2} = \cos^2\theta$

* $\cos 2\theta = 1 - 2\sin^2\theta$

$\Rightarrow \cos 2\theta - 1 = -2\sin^2\theta$

$\Rightarrow -(1 - \cos 2\theta) = -2\sin^2\theta$

$\Rightarrow 1 - \cos 2\theta = 2\sin^2\theta$

$\Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$

Imp. find $L\{\sin^2 3t\}$

Soln. $L\{\sin^2 3t\}$

$= L\left\{\frac{1-\cos 6t}{2}\right\}$

$= \frac{1}{2} L\{1-\cos 6t\}$

$= \frac{1}{2}\left[L\{1\} - L\{\cos 6t\}\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$

$= \frac{1}{2}\left[\frac{s^2+36-s^2}{s(s^2+36)}\right]$

$= \frac{1}{2}\left[\frac{\cancel{36}\ 18}{s(s^2+36)}\right]$

$= \frac{18}{s(s^2+36)}$

IMP $L\{\sin^3 at\}$

$= L\{\sin^3 at\}$

$= L\left\{\dfrac{3\sin at - \sin 3at}{4}\right\}$

$= \frac{1}{4} L\{3\sin at - \sin 3at\}$

$= \frac{1}{4}\left[3L\{\sin at\} - L\{\sin 3at\}\right]$

$= \frac{1}{4}\left[\dfrac{3a}{s^2+a^2} - \dfrac{3a}{s^2+9a^2}\right]$

$= \frac{1}{4}\left[\dfrac{3a}{s^2+a^2} - \dfrac{3a}{s^2+9a^2}\right]$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow \sin 3\theta - 3\sin\theta = -4\sin^3\theta$

$\Rightarrow +(3\sin\theta - \sin 3\theta) = +4\sin^3\theta$

$\Rightarrow \dfrac{3\sin\theta - \sin 3\theta}{4} = \sin^3\theta$

Q) $L\{\cos^3 2t\}$

~~$= 4\cos\theta - 3\cos^3\theta$~~

Q) $L\{\cos^3 2t\}$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$4\cos^3\theta = \cos 3\theta + 3\cos\theta$

$\cos^3\theta = \dfrac{\cos 3\theta + 3\cos\theta}{4}$

$= L\left\{\dfrac{3\cos 2t + \cos 3\cdot 2t}{4}\right\}$

$= L\left\{\dfrac{3\cos 2t + \cos 6t}{4}\right\}$

$= \frac{1}{4} L\{3\cos 2t + \cos 6t\}$

$= \frac{1}{4}\left[3L\{\cos 2t\} + L\{\cos 6t\}\right]$

$= \frac{1}{4}\left[3 \times \dfrac{s}{s^2+2^2} + \dfrac{s}{s^2+6^2}\right]$

$= \frac{1}{4}\left[\dfrac{3s}{s^2+4} + \dfrac{s}{s^2+36}\right]$

First shifting thm:-

If $L\{f(t)\} = f(s)$

$\Rightarrow L\{e^{at} f(t)\} = f(s-a), \ s-a>0$

Ex-1

(1) $L\{2\} = 2/s$

$L\{e^{at} \cdot 2\} = \frac{2}{s-a}$

(2) $L\{\sin 2t\}$

$= \frac{2}{s^2+4}$

$L\{e^{t} \cdot \sin 2t\}$

$= \frac{2}{(s-1)^2+4}$

$= \frac{2}{s^2-2s+1+4}$

$= \frac{2}{s^2-2s+5}$

IMP

Ex-1

Find the laplace of $e^{2t} \cdot t^5$

$= L\{t^5\} = \frac{5!}{s^{5+1}}$

$= \frac{120}{s^6}$

$L\{e^{2t} \cdot t^5\}$

$= \frac{120}{(s-2)^6}$

Ex-2 $L\{e^{at} \cdot \cos at\}$

$L\{\cos at\}$

$= \frac{s}{s^2+a^2}$

$\therefore L\{e^{at} \cdot \cos at\}$

$= \frac{s-a}{(s-a)^2+a^2}$

$= \frac{s-a}{s^2+a^2-2sa+a^2} = \frac{s-a}{2a^2+s^2-2sa}$

Ex-1 $L\{e^{3t}\cdot \sin^2 t\}$

$$= L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2} L\{1-\cos 2t\}$$

$$= \frac{1}{2} L\{1\} - L\{\cos 2t\}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s^2}+4-\cancel{s^2}}{s(s^2+4)}\right]$$

$$= \frac{1}{2}\left[\frac{4}{s(s^2+4)}\right]$$

$$= \frac{\cancel{2}}{\cancel{(s^3+s)}} \quad \frac{2}{s^3+4s}$$

$$\therefore L\{e^{3t}\cdot \sin^2 t\} = \frac{1}{2\left[(s-3)^3+(s-3)\right]}$$

$$= \frac{1}{2(s^3-27+9s^2-27s+s-3)}$$

$$= \frac{1}{2(s^3-9s^2-26s-30)}$$

* $L\{e^{-t}\sin 4t\}$

$$= L\{\sin 4t\}$$

$$= \frac{4}{s^2+16}$$

$$= L\{e^{-t}\sin 4t\}$$

$$= \frac{4}{(s+1)^2+16}$$

(i) $e^{2t} t^5$

$= L\{e^{2t} t^5\}$

$= L\{t^5\}$

$= \frac{5!}{s^{5+1}}$

$= \frac{120}{s^6}$

$L\{e^{2t} \cdot t^5\}$

$= \frac{120}{(s-2)^6}$

(ii) $e^{-t}(3\sinh 2t - 2\cosh 3t)$

$= L\{3\sinh 2t - 2\cosh 3t\}$

$= L\{3\sinh 2t\} - 2L\{\cosh 3t\}$

$= 3L\{\sinh 2t\} - 2L\{\cosh 3t\}$

$= 3 \times \frac{2}{s^2-4} - 2 \times \frac{s}{s^2+9}$

$= \frac{6}{s^2-4} - \frac{2s}{s^2+9}$

$= L\, e^{-t}(3\sinh 2t - 2\cosh 3t)$

$\cancel{= \frac{12}{s^2-4} - \frac{2s}{s^2+9}}$

$= \frac{6}{(s+1)^2-4} - \frac{2(s+1)}{(s+1)^2+9}$

$= \frac{6}{s^2+2s+1^2-4} - \frac{2s+2}{s^2+2s+1^2+9}$

$= \frac{6}{s^2+2s+1-4} - \frac{2s+2}{s^2+2s+1+9}$

$= \frac{6}{s^2+2s-3} - \frac{2s+2}{s^2+2s+10}$

(iii) $e^{3t} \sin^2 t$

$= L\{e^{3t} \sin^2 t\}$

$= L\{\sin^2 t\}$

$= L\left\{\frac{1-\cos t}{2}\right\}$

$= \frac{1}{2} L\{1-\cos t\}$

$= \frac{1}{2} L\{1\} - L\{\cos t\}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$

$= \frac{1}{2}\left[\frac{s^2+1-s^2}{s(s^2+1)}\right]$

$= \frac{1}{2}\left[\frac{1}{s(s^2+1)}\right]$

$= \frac{1}{2}\left[\frac{1}{(s-3)\,[(s-3)^2+1]}\right] = \frac{1}{2(s^3+s)}$

$= L\left\{ e^{3t} \sin^2 t \right\}$

$= \dfrac{1}{2\left[(s-3)^3 + (s-3)\right]}$

$= \dfrac{1}{2\left[s^3 + 9s^2 - 27s - 27 + s - 3\right]}$

$= \dfrac{1}{2\left[s^3 + 9s^2 - 26s + 30\right]}$

(iv) $e^{-4t} \cos t \sin 2t$

$= L\{\cos t . \sin 2t\}$

$= \frac{1}{2} L\{\sin(A+B) + \sin(A-B)\}$

$= \frac{1}{2} L\{\sin(t+2t) + \sin(t-2t)\}$

$= \frac{1}{2} L\{\sin 3t + \sin(-t)\}$

$= \frac{1}{2} L\{\sin 3t - \sin t\}$

$L \, \frac{1}{2}\{\sin 3t - \sin t\}$

$= \frac{1}{2} L\{\sin 3t - \sin t\}$

$= \frac{1}{2} \left[L\{\sin 3t\} - L\{\sin t\}\right]$

$= \frac{1}{2}\left[\dfrac{3}{s^2+9} - \dfrac{1}{s^2+1}\right]$

$= \frac{1}{2}\left(\dfrac{3(s^2+1) - 1(s^2+9)}{(s^2+9)(s^2+1)}\right)$

$= \frac{1}{2}\left[\dfrac{3s^2 + 3 - s^2 - 9}{(s^2+9)(s^2+1)}\right]$

$= \frac{1}{2}\left[\dfrac{2s^2 - 6}{(s^2+9)(s^2+1)}\right]$

$= \frac{1}{2}\left[\dfrac{2(s^2-3)}{(s^2+9)(s^2+1)}\right]$

$= \dfrac{s^2-3}{(s^2+9)(s^2+1)}$

$= \left(\dfrac{(s+4)^2 - 3}{\left[(s+4)^2+9\right]\left[(s+4)^2+1\right]}\right)$

* Transform of $[t^n f(t)]$, $n = 1, 2 \ldots \infty$

If $L\{f(t)\}$

$= f(s)$

$$\boxed{\Rightarrow L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)}$$

e.g.:- $L\{t f(t)\} = (-1)^1 \frac{d}{ds} f(s)$

$= -\frac{d}{ds} f(s)$

$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} f(s)$

$= \frac{d^2}{ds^2} f(s)$

e.g. $- L\{t \cdot \sin at\}$

$= L\{\sin at\} = \frac{a}{a^2+s^2} = \frac{a}{s^2+a^2}$

$L\{t \cdot \sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$

$= -\left[\frac{(s^2+a^2)\frac{d}{ds}(a) - a\frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2}\right]$

$= -\left[\frac{-a \cdot 2s}{(s^2+a^2)^2}\right]$

$= \frac{2as}{(s^2+a^2)^2}$

Ex-2

$L\{t.e^{-t}.\sin 4t\}$

$= L\{\sin 4t\}$

$= \frac{4}{s^2+16}$

$= L\{t.\sin 4t\}$

$= -\frac{d}{ds}\left[\frac{4}{s^2+16}\right]$

$= -\left[\frac{(s^2+16)\frac{d}{ds}4 - 4\frac{d}{ds}(s^2+16)}{(s^2+16)^2}\right]$

$= -\left[\frac{-4\cdot 2s}{(s^2+16)^2}\right]$

$= \left(\frac{8s}{s^4+256+32s^2}\right)$

$\therefore L\left\{e^{-t}\,t\sin 4t\right\}$

$= \frac{8(s+1)}{(s+1)^4+256+32(s+1)^2}$

$* L\{t\,e^{2t}.\sin 2t\}$

$= L\{\sin 2t\}$

$= \frac{2}{s^2+4}$

$L\{t\sin 2t\}$

$= -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$

$= -\left[\frac{(s^2+4)\frac{d}{ds}2 - 2\frac{d}{ds}(s^2+4)}{(s^2+4)^2}\right]$

$= -\left[\frac{-2\times 2s}{(s^2+4)^2}\right]$

$= \frac{4s}{(s^2+4)^2}$

$$L\{t \cdot e^{-2t} \cdot \sin 2t\}$$

$$= \frac{4(s+2)}{[(s+2)^2+4]^2}$$

$$= \frac{4s+8}{(s^2+4s+4+4)^2}$$

$$= \frac{4s+8}{(s^2+4s+8)^2}$$

$$= \frac{4s+8}{(s^2+4s+8)^2}$$

$$* \; L\{t \cdot \sin 3t \cdot \cos 2t\}$$

$$= L\{\sin 3t \cdot \cos 2t\} \qquad \sin A+B + \sin(A-B)$$

$$= L\{\tfrac{1}{2}\sin 5t + \sin t\}$$

$$= \tfrac{1}{2} L\{\sin 5t + \sin t\}$$

$$= \tfrac{1}{2}\left[L\{\sin 5t\} + L\{\sin t\}\right]$$

$$= \tfrac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$\therefore L\{t \sin 3t \cdot \cos 2t\}$$

$$= -\frac{d}{ds}\left\{\tfrac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]\right\}$$

$$\cancel{= \frac{d}{ds} \tfrac{1}{2}\{}$$

$$= -\tfrac{1}{2}\left[\frac{d}{ds}\frac{5}{s^2+25} + \frac{d}{ds}\frac{1}{s^2+1}\right]$$

$$= -\frac{1}{2}\left[\frac{(s^2+25)\frac{d}{ds}5 - 5\frac{d}{ds}(s^2+25)}{(s^2+25)^2} + \frac{(s^2+1)\frac{d}{ds}1 - 1\frac{d}{ds}(s^2+1)}{(s^2+1)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{-5\times 2s}{(s^2+25)^2} + \frac{-1\times 2s}{(s^2+1)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{-10s}{(s^2+25)^2} + \frac{-2s}{(s^2+1)^2}\right]$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

* $L\{t.e^{-t}\cos ht\}$

* $L\{t\sin^2 t\}$

* $L\{t^2.e^{-2t}\cos t\}$

* $L\{t.e^{-t}\cos ht\}$

$= L\{\cos ht\}$

$= \frac{s}{s^2+1^2}$

$= \frac{s}{s^2+1}$

$= L\{t\cdot\cos ht\}$

$= -\frac{d}{ds}\left\{\frac{s}{s^2+1}\right\}$

$$= \frac{(s^2+1)\frac{d}{ds}s - s\frac{d}{ds}(s^2+1)}{(s^2+1)^2}$$

$$= \frac{s^2+1-s\times 2s}{(s^2+1)^2} = \frac{s^2+1-2s^2}{(s^2+1)^2} = -\frac{-s^2+1}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

$\therefore L\{t.e^{-t}\cos ht\}$

$$= \frac{(s+1)^2-1}{[(s+1)^2+1]^2}$$

$$* \mathcal{L}\left\{t \cdot \sin^2 t\right\}$$

$$= \mathcal{L}\left\{t \cdot \frac{1-\cos 2t}{2}\right\}$$

$$= \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{1-\cos 2t\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{1\} - \mathcal{L}\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+1-s^2}{s(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s(s^2+1)}\right]$$

$$\mathcal{L}\left\{t \cdot \sin^2 t\right\}$$

$$= \frac{1}{2}\frac{d}{ds}\left\{\frac{1}{s(s^2+1)}\right\}$$

$$= \frac{1}{2}\,\frac{s(s^2+1)\frac{d}{ds}1 - 1\frac{d}{ds}s(s^2+1)}{\left[s(s^2+1)\right]^2}$$

$$= \cancel{s(s^2+1)} - \frac{1}{2}\,\frac{s(s^2+1)}{\left[s(s^2+1)\right]^2}$$

$$= \frac{1}{2}\left[t \cdot \sin^2 t\right]$$

$$= \frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{d}{ds}\frac{1}{s} - \frac{d}{ds}\frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{s^2} - \frac{(s^2+4)\frac{d}{ds}s - s\frac{d}{ds}(s^2+4)}{(s^2+4)^2}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{s^2} - \frac{(s^2+4) - s \cdot 2s}{(s^2+4)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{s^2} + \frac{(s^2+4) - 2s^2}{(s^2+1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{-s^2+1}{(s^2+1)^2}\right]$$

Transform of $\frac{1}{t}\cdot f(t)$

If $L[f(t)] = f(s)$

$$\Rightarrow L\left[\frac{1}{t}\cdot f(t)\right] = \int_s^\infty e^{-st} f(s)\, ds = \int_s^\infty f(s)\, ds$$

Ex-1 $\quad L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$

$= L\left\{e^{-at} - e^{-bt}\right\}$

$= L\left\{e^{-at}\right\} - L\left\{e^{-bt}\right\}$

$= \frac{1}{s+a} - \frac{1}{s+b}$

$\therefore L\left\{\frac{1}{t}\left(e^{-at} - e^{-bt}\right)\right\}$

$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$

$= \int_s^\infty \frac{1}{s+a} ds - \int_s^\infty \frac{1}{s+b} ds$

$= \left[\log(s+a)\right]_s^\infty - \left[\log(s+b)\right]_s^\infty$

$= (\log\infty - \log(s+a)) - (\log\infty - \log(s+b))$

$= -\log(s+a) + \log(s+b)$

$= \log(s+b) - \log(s+a)$

$= \log\frac{s+b}{s+a}$

Ex-2

Find $L\left\{\frac{\cos 2t - \cos 3t}{t}\right\}$

Soln $L\{\cos 2t - \cos 3t\}$

$= L\{\cos 2t\} - L\{\cos 3t\}$

$= \frac{s}{s^2+4} - \frac{s}{s^2+9}$

$\therefore L\left\{\frac{1}{t}(\cos 2t - \cos 3t)\right\}$

$= \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$

$= \int_s^\infty \frac{s}{s^2+4}\,ds - \int_s^\infty \frac{s}{s^2+9}\,ds$ $\left\{\text{let } t = s^2+4,\ \frac{dt}{ds} = 2s \Rightarrow \frac{1}{2}\frac{dt}{ds} = s \Rightarrow \frac{1}{2}dt = s\,ds\right\}$

when $s=\infty$, $t = \infty^2+4 = \infty$; when s, $t = s^2+4$

$= \int_s^\infty \frac{\frac{1}{2}dt}{t} - \int_s^\infty \frac{\frac{1}{2}dv}{v}$

$= \int_s^\infty \frac{1}{2}\frac{dt}{t} - \int_s^\infty \frac{1}{2}\frac{dv}{v}$ $\left\{v = s^2+9,\ \frac{dv}{ds} = 2s,\ \frac{1}{2}\frac{dv}{ds} = s,\ \frac{1}{2}dv = s\,ds\right\}$

$= \frac{1}{2}[\log t]_s^\infty - \frac{1}{2}[\log v]_s^\infty$

$= \frac{1}{2}[\log(s^2+4)]_s^\infty - \frac{1}{2}[\log(s^2+9)]_s^\infty$

$= \frac{1}{2}(\log\infty - \log(s^2+4)) - \frac{1}{2}(\log\infty - \log(s^2+9))$

$= -\frac{1}{2}(\log(s^2+4)) + \frac{1}{2}(\log(s^2+9))$

$= \frac{1}{2}[-\log(s^2+4) + \log(s^2+9)]$

$= \frac{1}{2}\left[\log\frac{(s^2+9)}{(s^2+4)}\right]$

Imp.
Ex-3

$$L\left\{\frac{1-e^{t}}{t}\right\}$$

$$= L\{1-e^{t}\}$$

$$= L\{1\}-L\{e^{t}\}$$

$$= \frac{1}{s}-\frac{1}{s-1}$$

$$L\left\{\frac{1}{t}(1-e^{t})\right\}$$

$$= L\left\{\frac{1}{t}\left(\frac{1}{s}-\frac{1}{s-1}\right)\right\}$$

$$= \int_s^{\infty}\frac{1}{s}\,ds-\int_s^{\infty}\frac{1}{s-1}\,ds$$

$$= [\log s]_s^{\infty}-[\log(s-1)]_s^{\infty}$$

$$= (\log\infty-\log s)-(\log\infty-\log(s-1))$$

$$= -\log s+\log(s-1)$$

$$= \log\frac{(s-1)}{s}$$

Find $L\left\{\frac{\sin t}{t}\right\}$

$$= L\{\sin at\}$$

$$= \frac{a}{s^2+a^2}$$

$$\therefore L\left\{\frac{1}{t}\sin at\right\}$$

$$= \int_s^{\infty}\left(\frac{a}{s^2+a^2}\right)ds$$

$$= \int_s^{\infty}\frac{a}{a^2+s^2}\,ds$$

$$= \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^{\infty}$$

$$= \tan^{-1}\infty-\tan^{-1}(s/a)$$

$$= \frac{\pi}{2}-\tan^{-1}(s/a)$$

$$= \cot^{-1}(s/a)$$

~~(2) $L\{\cos at\}$~~

(1) $L\left\{\frac{\cos at-\cos bt}{t}\right\}$

(2) $L\left\{\frac{1-\cos 2t}{t}\right\}$

(3) $L\left\{\frac{1-e^{at}}{t}\right\}$

(4) $L\left\{\frac{\sin^2 t}{t}\right\}$

* Find $l\left\{\frac{e^{-t}\sin t}{t}\right\}$

$= l\left\{e^{-t}\sin t\right\}$

$= l\left\{\sin t\right\}$

$= \frac{1}{s^2+1}$

$l\left\{e^{-t}\sin t\right\}$

$= \frac{1}{(s+1)^2+1}$

$\therefore l\left\{\frac{1}{t}e^{-t}.\sin t\right\}$

$= \int_s^\infty \frac{1}{(s+1)^2+1}ds$

$= \left[\tan^{-1}(s+1)\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}(s+1)$

$= \pi/2 - \tan^{-1}(s+1)$

$= \cot^{-1}(s+1)$

* $l\left\{\frac{\cos at - \cos bt}{t}\right\}$

$l\left\{\cos at - \cos bt\right\}$

$= l\left\{\cos at\right\} - l\left\{\cos bt\right\}$

$= \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)$

$= l\ \frac{1}{t}\left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\}$

$= \frac{1}{t}\left\{\frac{s}{s^2+a^2}\right\} - \frac{1}{t}\left\{\frac{s}{s^2+b^2}\right\}$

$= \cancel{\frac{(s^2-a^2)}{(s^2-a^2)^2}}$

$= \int_s^\infty \frac{s}{s^2+a^2}ds - \int_s^\infty \frac{s}{s^2+b^2}ds$

$= \left[\frac{\frac{dt}{2}}{t}\right]_s^\infty - \left[\frac{\frac{dv}{2}}{v}\right]_s^\infty$

$= \frac{1}{2}\left[\frac{dt}{t}\right]_s^\infty - \frac{1}{2}\left[\frac{dv}{v}\right]_s^\infty$

$= \frac{1}{2}\left[\frac{dt}{s^2+a^2}\right]_s^\infty - \frac{1}{2}\left[\frac{dv}{s^2+b^2}\right]_s^\infty$

$\boxed{v = s^2-b^2}$

$t = s^2+a^2$

$\frac{dt}{ds} = \frac{d}{ds}s^2 + \frac{d}{ds}a^2$

$\frac{dt}{ds} = 2s$

$dt = 2s \times ds$

$\frac{dt}{2} = s\,ds$

$= \frac{1}{2}dt = s\,ds$

$$= \frac{1}{2}\left[\log(s^2+a^2)\right]_s^\infty - \frac{1}{2}\left[\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\infty - \log(s^2+a^2)\right] - \frac{1}{2}\left[\log\infty - \log(s^2+b^2)\right]$$

$$= \frac{1}{2}\left[-\log(s^2+a^2) + \log(s^2+b^2)\right]$$

$$= \frac{1}{2}\log\frac{(s^2+b^2)}{(s^2+a^2)}$$

$$= \frac{1}{2}\left[\log\frac{(s^2+b^2)}{(s^2+a^2)}\right]$$

$$* \; L\left\{\frac{1-\cos 2t}{t}\right\}$$

$$= L\{1-\cos 2t\}$$

$$= L\{1\} - L\{\cos 2t\}$$

$$= \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{\frac{1}{t}\, 1-\cos 2t\right\}$$

$$= L\left\{\frac{1}{t}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right\}$$

$$\cancel{= \frac{1}{t}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}}$$

$$= \int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{s}{s^2+4}\,ds$$

$$= \left[\log s\right]_s^\infty - \frac{1}{2}\left[\frac{dt}{t}\right]_s^\infty$$

$$= \left[\log\infty - \log s\right] - \frac{1}{2}\left[\frac{dt}{s^2+4}\right]_s^\infty$$

$$= \left[\infty - \log s\right] - \frac{1}{2}\left[\log s^2+4\right]_s^\infty$$

$$t = s^2+4$$

$$\frac{dt}{ds} = \frac{d}{ds}s^2 + \frac{d}{ds}4$$

$$\frac{dt}{ds} = 2s$$

$$dt = 2s\,ds$$

$$\frac{dt}{2} = s\,ds$$

$$\frac{1}{2}dt = s\,ds$$

$$= \left[-\log s\right] - \frac{1}{2}\left[\log\infty - \log(s^2+4)\right]$$

$$= -\log s - \frac{1}{2}\left\{-\log(s^2+4)\right\}$$

$$= -\log s + \frac{1}{2}\left\{\log(s^2+4)\right\}$$

$$= \frac{1}{2}\frac{\log(s^2+4)}{s}$$

* $\mathcal{L}\left\{\frac{1-e^{at}}{t}\right\}$

$= \mathcal{L}\left\{1-e^{at}\right\}$

$= \mathcal{L}\{1\} - \mathcal{L}\{e^{at}\}$

$= \frac{1}{s} - \frac{\cancel{e^{at}}\, 1}{s-a}$

$= \frac{1}{s} - \frac{1}{s-a}$

$\mathcal{L}\left\{\frac{1}{t}\cdot(1-e^{at})\right\}$

$= \int_s^\infty 1-e^{at}$

$= \int_s^\infty \frac{1}{s} - \frac{1}{s-a}$

$= \int_s^\infty \frac{1}{s} - \int_s^\infty \frac{1}{s-a}$

$= \left[\log s\right]_s^\infty - \left[\log(s-a)\right]_s^\infty$

$= (\log\infty - \log s) - (\log\infty - \log(s-a))$

$= -\log s - (-\log(s-a))$

$= -\log s + \log(s-a)$

$= \log\frac{(s-a)}{s}$

* $\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\}$

$= \mathcal{L}\{\sin^2 t\}$

$= \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\}$

$= \frac{1}{2}\mathcal{L}\{1-\cos 2t\}$

$= \frac{1}{2}\mathcal{L}\{1\} - \mathcal{L}\{\cos 2t\}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2-4}\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$= \frac{1}{2}\mathcal{L}\left\{\frac{1}{t}\cdot\sin^2 t\right\}$

$= \frac{1}{2}\int_s^\infty\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$= \frac{1}{2}\int_s^\infty \frac{1}{s}ds - \int_s^\infty \frac{s}{s^2+4}ds$

$= \frac{1}{2}\left[\log s\right]_s^\infty - \frac{1}{2}\left[\frac{dt}{t}\right]$

$= \frac{1}{2}\left[\log\infty - \log s\right] - \frac{1}{2}\left[\frac{dt}{s^2+4}\right]_s^\infty$

$= \frac{1}{2}\left[-\log s\right] - \frac{1}{2}\left[\log(s^2+4)\right]_s^\infty$

$t = s^2 - 1$

$\frac{dt}{ds} = 2s$

$dt = 2s\,ds$

$\frac{1}{2}dt = s\,ds$

$= \frac{1}{2}[-\log s] - \frac{1}{2}[\log \infty - \log(s^2+4)]$

$= \frac{1}{2}[-\log s] - \frac{1}{2}[-\log(s^2+4)]$

$\cancel{= \frac{1}{2}[-\log s - \log(s^2+4)]}$ $= \frac{1}{2}\left[\frac{1}{2}\log(s^2+4) - \log s\right]$

$\cancel{= -\frac{1}{2}[\log s + \log(s^2+4)]}$ $= \frac{1}{4}\log\frac{(s^2+4)}{s}$

$\cancel{= -\frac{1}{2}\log(s^2+4)(s)}$.

<u>change of Scale property :-</u>

By Def^n of Laplace $L\{f(t)\} = \bar{f}(s)$

$$\boxed{L\{f(t)\} = \int_0^\infty e^{-st} f(s)\, ds}$$

* $\int_0^\infty t \cdot e^{-3t} \cdot \sin t \, dt$

<u>Solution</u> $\int_0^\infty t \cdot e^{-3t} \sin t\, dt$ (Have $s=3$)

$= L\{\sin t\}$

$= \frac{1}{s^2+1}$

$= L\{t \sin t\}$

$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$= -\frac{(s^2+1)\frac{d}{ds}1 - 1\frac{d}{ds}(s^2+1)}{(s^2+1)^2}$

$= -\frac{\cancel{(s^2+1)} - 1 \times 2s}{(s^2+1)^2}$

$= \frac{2s}{(s^2+1)^2}$

$\int_0^\infty t \cdot e^{-3t} \cdot \sin t\, dt$

$= \frac{2s}{(s^2+1)^2}$

$= \frac{2 \times 3}{(3^2+1)^2}$

$= \frac{6}{(9+1)^2} = \frac{6}{100}$

$= 3/50$

* $\int_0^\infty t \cdot e^{-2t} \cdot \cos t \, dt$

$= L\{\cos t\}$

$= \frac{s}{s^2+1}$

$= L\{t \cos t\}$

$= -\frac{d}{ds} \frac{s}{s^2+1}$

$= -\frac{(s^2+1)\frac{d}{ds}s - s\frac{d}{ds}(s^2+1)}{(s^2+1)^2}$

$= -\frac{(s^2+1) - s \times 2s}{(s^2+1)^2}$

$= -\frac{(s^2+1) - 2s^2}{(s^2+1)^2}$

$= -\frac{(s^2+1-2s^2)}{(s^2+1)^2}$

$= -\frac{(-s^2+1)}{(s^2+1)^2}$

$= \frac{s^2-1}{(s^2+1)^2}$

$\therefore \int_0^\infty t \cdot e^{-2t} \cdot \cos t \, dt$

$= \frac{s^2-1}{(s^2+1)^2}$

$= \frac{(-2)^2-1}{((-2)^2+1)^2}$

$= \frac{4-1}{(4+1)^2} = \frac{3}{(5)^2}$

$= \frac{3}{25}$

Inverse laplace

By definiation of laplace

$$\mathcal{L}\{f(t)\} = f(s)$$

$$\boxed{\Rightarrow f(t) = \mathcal{L}^{-1} f(s)}$$

e.g. – $\mathcal{L}\{1\} = 1/s$

$$\Rightarrow \mathcal{L}^{-1}(1/s) = 1$$

e.g. – $\mathcal{L}\{t\} = \frac{1}{s^2}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

* $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$$

Inverse Formula

1. $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
2. $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$
3. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
4. $\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$
5. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
6. $\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$
7. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
8. $\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$

* $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

* $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

* (i) $L^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t$

(ii) $L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$

Ex-1

Find inverse laplace of followings:-

(i) $\frac{3}{s+3}$

Solun = $L^{-1}\left\{\frac{3}{s+3}\right\}$

$= 3L^{-1}\left\{\frac{1}{s+3}\right\}$

$= 3e^{-3t}$

$= 3e^{-3t}$

(ii) $\frac{2s}{s^2+9}$

$= 2L^{-1}\left\{\frac{s}{s^2+9}\right\}$

$= 2\cos 3t$

$= 2\cos 3t$

(iii) $\frac{2(s-1)}{s^2+8} = \frac{2s}{s^2+8} - \frac{1}{s^2+8}$

$\cancel{= 2L^{-1}\left\{\frac{s-1}{s^2+8}\right\}} = 2L^{-1}\left\{\frac{s}{s^2+\sqrt{8}^2}\right\} - L^{-1}\left\{\frac{1}{s^2+\sqrt{8}^2}\right\}$

$= 2\times\cos\sqrt{8}t - \frac{1}{\sqrt{8}}\sin\sqrt{8}t$

(iv) $\dfrac{3s-4}{16-s^2}$

$$= \frac{3s}{16-s^2} - \frac{4}{16-s^2}$$

$$= -3L^{-1}\left\{\frac{s}{s^2-16}\right\} - L^{-1}\left\{\frac{4}{s^2-16}\right\}$$

$$= -3 \times \cosh 4t + \sinh 4t$$

$$= -3\cosh 4t + \sinh 4t$$

(v) $L^{-1}\left\{\dfrac{3s}{s^2+2s-8}\right\}$

$$= 3L^{-1}\left\{\frac{s}{(s^2+2s+1^2)-9}\right\}$$

$$= 3L^{-1}\left\{\frac{s}{(s+1)^2-3^2}\right\}$$

$$= 3L^{-1}\left\{\frac{s+1-1}{(s+1)^2-3^2}\right\}$$

$$= 3\left[L^{-1}\left\{\frac{s+1}{(s+1)^2-3^2}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2-3^2}\right\}\right]$$

$$= 3\left[e^{-t}\cosh 3t - \frac{1}{3}e^{-t}\sinh 3t\right]$$

$$* \; L^{-1}\left\{\frac{3s-7}{s^2-2s-3}\right\}$$

$$= L^{-1}\left\{\frac{3s}{s^2-2s-3}\right\} - L^{-1}\left\{\frac{7}{s^2-2s-3}\right\}$$

$$= L^{-1}\,3\left\{\frac{s}{(s^2-2s\cdot1+1^2)-4}\right\} - 7L^{-1}\left\{\frac{1}{(s^2-2\cdot s\cdot 1+1^2)-4}\right\}$$

$$= 3L^{-1}\left\{\frac{(s-1)+1}{(s-1)^2-(2)^2}\right\} - 7\;L^{-1}\left\{\frac{1}{(s-1)^2-(2)^2}\right\}$$

$$= \cancel{3\times}\left(e^{t}\cosh 2t - 7e^{t}\frac{1}{2}\sinh 2t\right)$$

$$= \cancel{3e^{t}\cosh 3A} - \frac{7}{2}e^{t}\,\cancel{3}\sinh 2t$$

$$= 3\times L^{-1}\left\{\frac{s-1}{(s-1)^2-2^2}\right\} + L^{-1}\left\{\frac{1}{(s-1)^2-2^2}\right\} - \frac{7}{2}e^{t}\sinh 2t$$

$$= 3\times e^{t}\cosh 2t + \frac{1}{2}\sinh 2t - \frac{7}{2}e^{t}\sinh 2t$$

$$= 3\left[e^{t}\cosh 2t + \frac{1}{2}e^{t}\sinh 2t\right] - \frac{7}{2}e^{t}\sinh 2t$$

$$= 3\left[e^{t}\cosh 2t + \frac{1}{2}e^{t}\sinh 2t\right] - \frac{7}{2}e^{t}\sinh 2t$$

Inverse laplace by partial fraction :-

Case-1

If the denominator is lineare but not reapeated.

* $L^{-1}\left\{\frac{1}{(S-1)(S-2)}\right\}$

$= \frac{A}{S-1} + \frac{B}{S-2}$

Case-2

If the denominator is lineare but reapeated.

* $L^{-1}\left\{\frac{1}{(S-a)^2}\right\}$

$= \frac{A}{S-a} + \frac{B}{(S-a)^2} + \frac{C}{(S-a)^3} +$, so on.........

Case-3

If the denominator is quadratic but non reapeated.

* $L^{-1}\left\{\frac{1}{(s^2-a)(s^2-b)}\right\}$

$= \frac{Ax+B}{s^2-a} + \frac{Cx+D}{s^2-b} + \ldots\ldots$

Case-4

If the denominator is quadratic but reapeated.

* $L^{-1}\left\{\frac{1}{(s^2-a)^2(s^2-b)}\right\} = \frac{Ax+B}{s^2-a} + \frac{Cx+D}{(s^2-a)^2} + \frac{Ex+F}{s^2-b}$

* Find $L^{-1}\left\{\frac{s^2+s-2}{s(s+3)(s-2)}\right\}$

$$= \frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$= \frac{A(s+3)(s-2)+Bs(s-2)+Cs(s+3)}{s(s+3)(s-2)}$$

$$\therefore s^2+s-2 = A(s+3)(s-2)+Bs(s-2)+Cs(s+3) \quad \text{---}(1)$$

$\therefore$ $\boxed{s=0}$ put in equation(1)-

$\Rightarrow s^2+s-2 = A(s+3)(s-2)+Bs(s-2)+Cs(s+3)$

$\Rightarrow -2 = A(-6)+0+0$

$\Rightarrow +2 = +6A$

$\Rightarrow A = \cancel{2}/\cancel{6}_3 = 1/3$

$\boxed{\begin{array}{l} s+3=0 \\ s=-3 \end{array}}$ Put in equation (1) :-

$\Rightarrow s^2+s-2 = A(s+3)(s-2)+Bs(s-2)+Cs(s+3)$

$\Rightarrow 9-3-2 = \cancel{A(+3+2)}\ 0 + B\,-3(-3-2)$

$\Rightarrow 4 = +B.-3(-5)$

$\Rightarrow 4 = B(15)$

$\Rightarrow B = 4/15$

$\boxed{\begin{array}{l} s-2=0 \\ s=2 \end{array}}$ put in equation (1) :-

$\Rightarrow s^2+s-2 = A(s+3)(s-2)+Bs(s-2)+Cs(s+3)$

$\Rightarrow 4+\cancel{2}-\cancel{2} = C\,2(2+3)$

$\Rightarrow y = C\,(2\times5)$

$\Rightarrow y = C\times10$

$\Rightarrow C = \cancel{4}/\cancel{10}\,{}^{2}{}_{5} = 2/5$

Hence, $\dfrac{s^2+s-2}{s(s+3)(s-2)} = \dfrac{1/3}{s} + \dfrac{4/15}{s+3} + \dfrac{2/5}{s-2}$

$\Rightarrow$ By taking inverse laplace transformation :-

$\Rightarrow L^{-1}\left\{\dfrac{s^2+s-2}{s(s+3)(s-2)}\right\} = L^{-1}\left\{\dfrac{1/3}{s}\right\} + L^{-1}\left\{\dfrac{4/15}{s+3}\right\} + L^{-1}\left\{\dfrac{2/5}{s-2}\right\}$

$= \frac{1}{3}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{15}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{2}{5}L^{-1}\left\{\frac{1}{s-2}\right\}$

$= \frac{1}{3}\times1 + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$

$= \frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$

* $L^{-1}\left\{\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}\right\}$

$= L^{-1}\left\{\dfrac{s^2-10s+13}{(s-7)(s^2-2s-3s+6)}\right\}$

$= L^{-1}\left\{\dfrac{s^2-10s+13}{(s-7)(s(s-2)-3(s-2))}\right\}$

$= L^{-1}\left\{\dfrac{s^2-10s+13}{(s-7)(s-2)(s-3)}\right\} = \dfrac{A}{s-7} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$= \dfrac{A(s-2)(s-3)+B(s-7)(s-3)+C(s-7)(s-2)}{(s-7)(s-2)(s-3)}$

$s^2-10s+13 = A(s-2)(s-3)+B(s-7)(s-3)+C(s-7)(s-2)$ ---- (i)

$$\boxed{\begin{array}{c} s-7=0 \\ s=7 \end{array}}$$

$$s^2-10s+13 = A(s-2)(s-3)+B(s-7)(s-3)+C(s-7)(s-2)$$

$$\Rightarrow 7^2-10\cdot 7+13 = A(7-2)(7-3)$$

$$\Rightarrow 49-70+13 = A(5)(4)$$

$$\Rightarrow 62-70 = A\times 20$$

$$\Rightarrow -8 = 20A$$

$$\Rightarrow \frac{-8}{20} = A$$

$$\Rightarrow A = -2/5$$

$$\boxed{\begin{array}{c} s-2=0 \\ s=2 \end{array}}$$

$$s^2-10s+13 = A(s-2)(s-3)+B(s-7)(s-3)+C(s-7)(s-2)$$

$$\Rightarrow 2^2-10\cdot 2+13 = B(2-7)(2-3)$$

$$\Rightarrow 4-20+13 = B(-5)(-1)$$

$$\Rightarrow -20+17 = B\times 5$$

$$\Rightarrow -3 = 5B$$

$$\Rightarrow B = -3/5$$

$$\boxed{\begin{array}{c} s-3=0 \\ s=3 \end{array}}$$

$$s^2-10s+13 = A(s-2)(s-3)+B(s-7)(s-3)+C(s-7)(s-2)$$

$$\Rightarrow 3^2-10\cdot 3+13 = C(s-7)(s-2)$$

$$\Rightarrow 9-30+13 = C(3-7)(3-2)$$

$$\Rightarrow 22-30 = C(-4)(1)$$

$\Rightarrow +8 = +C\times 4$

$\Rightarrow C = \cancel{8}/\cancel{4}^{\,2}$

$\Rightarrow C = 2$

Hence,

$$\cancel{\frac{s^2-10s+13}{(s-7)(s^2-5s+6)}} =$$

$$\frac{s^2-10s+13}{(s-7)(s-2)(s-3)} = \frac{-2/5}{s-7} + \frac{-3/5}{s-2} + \frac{2}{s-3}$$

By taking Inverse laplace transformation -

$$= L^{-1}\left\{\frac{-2/5}{s-7}\right\} + L^{-1}\left\{\frac{-3/5}{s-2}\right\} + L^{-1}\left\{\frac{2}{s-3}\right\}$$

$$= -2/5\, L^{-1}\left\{\frac{1}{s-7}\right\} + -3/5\, L^{-1}\left\{\frac{1}{s-2}\right\} + 2L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -2/5\, e^{7t} + -3/5\, e^{2t} + 2e^{3t}$$

$$= -2/5\, e^{7t} - 3/5\, e^{2t} + 2e^{3t}$$

* ~~Find~~ $\cancel{L^{-1}\left\{\frac{1+2s}{(s+2)^2(s^2-1)}\right\}}$

* Find $L^{-1}\left\{\frac{s}{(s-3)(s^2+4)}\right\}$

$$= \frac{s}{(s-3)(s^2+4)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4}$$

$$= \frac{A(s^2+4)+Bs+C(s-3)}{(s-3)(s^2+4)}$$

$$= A(s^2+4) + (Bs+C)(s-3) \quad \text{--------- (1)}$$

$s-3=0$

$s=3$ Put in equation (I) :-

$3 = A(9+4)$

$3 = A(13)$

$$\boxed{A = 3/13}$$

Comparing the coefficient of s^2 :-

$\Rightarrow 0 = A + B$

$\Rightarrow 0 = 3/13 + B$

$$\boxed{\Rightarrow B = -3/13}$$

$(Bs+C)(s-3)$
$= Bs^2 - 3Bs + Cs - 3C$

Comparing the coefficient of s :-

$\Rightarrow 1 = -3B + C$

$\Rightarrow 1 = -3 \times \frac{-3}{13} + C$

$\Rightarrow 1 = 9/13 + C$

$\Rightarrow C = \frac{1}{1} - \frac{9}{13}$

$\Rightarrow C = \frac{13-9}{13}$

$\Rightarrow C = 4/13$

Hence, $\frac{s}{(s-3)(s^2+4)} = \frac{3/13}{s-3} + \frac{-3/13\, s + 4/13}{s^2+4}$

$= L^{-1}\left(\frac{3/13}{s-3}\right) + L^{-1}\left\{\frac{-3/13\, s + 4/13}{s^2+4}\right\}$

$= 3/13\, L^{-1}\left\{\frac{1}{s-3}\right\} + L^{-1}\left\{\frac{\frac{-3}{13}s}{s^2+4}\right\} + L^{-1}\left\{\frac{4/13}{s^2+4}\right.$

$$= \frac{3}{13} e^{3t} + \frac{-3}{13} L^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{4}{13} L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{3}{13} e^{3t} + \frac{-3}{13} \cos 2t + \frac{\cancel{4}^{2}}{13} \frac{1}{\cancel{2}} \sin 2t$$

$$= \frac{3}{13} e^{3t} - \frac{3}{13} \cos 2t + \frac{2}{13} \sin 2t$$

(1) Find $L(e^{-2t}\sin t)$

$= L(\sin t)$

$= \frac{1}{s^2+1}$

$L(e^{-2t}\sin t)$

$= L\left(e^{-2t}\frac{1}{s^2+1}\right)$

$= \frac{1}{(s+2)^2+1}$

$= \frac{1}{s^2+4s+4+1}$

$= \frac{1}{s^2+4s+5}$

(2) Find $L^{-1}\frac{(3s+1)}{(s+1)(s^2+1)}$

$\frac{(3s+1)}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$

$= \frac{A(s^2+1)+(Bs+C)(s+1)}{(s+1)(s^2+1)}$

$= \frac{A(s^2+1)+(Bs+C)(s+1)}{(s+1)(s^2+1)}$

$\cancel{3s+1} = A(s^2+1)+(Bs+C)(s+1)$

$s+1=0$

$s=-1$

$3s+1 = A(s^2+1)+(Bs+C)(s+1)$

$\Rightarrow 3\cdot-1+1 = A(-1^2+1)$

$\Rightarrow -3+1 = A(1+1)$

$\Rightarrow -2 = A(2)$

$\Rightarrow A = -2/2$

$\Rightarrow A = -1$

comparing the coefficient of s^2:-

$\Rightarrow 0 = A+B$

$\Rightarrow -A = B$

$\Rightarrow B = -(-1)$

$\Rightarrow B = 1$

Comparing the coefficient of s:-

$\Rightarrow 3 = B+C$

$\Rightarrow C = 3-B$

$\Rightarrow C = 3-1$

$\Rightarrow C = 2.$

$$\frac{3s+1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$= \frac{-1}{s+1} + \frac{1s+2}{s^2+1}$$

$$L^{-1}\left\{\frac{3s+1}{(s+1)(s^2+1)}\right\} = L^{-1}\left\{\frac{-1}{s+1}\right\} + L^{-1}\left\{\frac{1s+2}{s^2+1}\right\}$$

$$= e^{-t} + L^{-1}\left\{\frac{1s}{s^2+1}\right\} + L^{-1}\left\{\frac{2}{s^2+1}\right\}$$

$$= e^{-t} + 1\, L^{-1}\left\{\frac{s}{s^2+1}\right\} + 2L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t} + 1 \times \cos 1t + 2 \times \sin 1t$$

$$= e^{-t} + \cos t + 2\sin t.$$

(3) $L^{-1}\left(\frac{e^{2t}\sin 3t}{t}\right)$

$$= L^{-1}(\sin 3t)$$

$$= \frac{3}{s^2+9}$$

$$L^{-1}\left(\frac{1}{t}\sin 3t\right)$$

$$= \int_s^\infty \sin 3t \, ds$$

$$= \int_s^\infty \frac{3}{s^2+9}\, ds$$

$$= \int_s^\infty \tan^{-1} s/3 \, ds$$

$$= \tan^{-1}\infty - \tan^{-1} s/3$$

$$= \pi/4 - \tan^{-1} s/3$$

$$= \cot^{-1} s/3.$$

Gamma function–

Def^n– Gamma function is a function of single real variable and it is denoted by $\Gamma(n)$. and defined as gamma of 'n'.

$$\boxed{\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx}$$

Properties

(i) $\Gamma(0) = \infty$

(ii) $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$

(iii) $\boxed{\Gamma(n+1) = n\Gamma(n)}$

e.g. $\Gamma(3) = \Gamma(2+1) = 2\Gamma(2)$

(iv) $\boxed{\Gamma(n+1) = n!}$

e.g. $\Gamma(3) = \Gamma(2+1) = 2! = 2 \times 1$
$= 2.$

Ex-1 $\Gamma(5/2)$

$= \Gamma(5/2)$

$= \Gamma(3/2 + 1)$

$= 3/2\, \Gamma(3/2)$

$= 3/2\, \Gamma(1/2 + 1)$

$= 3/2 \times 1/2\, \Gamma(1/2)$

$= 3/4\, \Gamma(1/2)$

$= 3/4 \times \sqrt{\pi}$

$= 3\sqrt{\pi}/4$

Ex-2 $\Gamma(4.5)$

$= \Gamma\left(\frac{45}{10}\right)$

$= \Gamma(7/2 + 1)$

$= 7/2\, \Gamma(7/2)$

$= 7/2 \times 5/2\, \Gamma(5/2)$

$= \frac{35}{4} \times 3/2\, \Gamma(3/2)$

$= \frac{105}{8}\, 1/2\, \Gamma(1/2)$

$= \frac{105}{16} \pi$

$= \frac{105\sqrt{\pi}}{16}$

$3/2 + 1/1$

$= \frac{3+2}{2} = 5/2$

$\frac{45}{10} - \frac{1}{1}$

$= \frac{45-10}{10}$

$= 35/10 + 1$

$= \frac{35+10}{10}$

$= \frac{45}{10}$

$7/2 - 1$

$= \frac{7-2}{2}$

$= 5/2$

$5/2 - 1$

$= \frac{5-2}{2}$

$= 3/2$

Ex-3 $\dfrac{\Gamma(7/3)}{\Gamma(4/3)}$

$\Gamma(7/3)$

$= \Gamma(4/3 + 1)$

$= 4/3\, \Gamma(4/3)$

$= 4/3\, \Gamma(1/3 + 1)$

$= 4/3 \times 1/3\, \Gamma(1/3)$

$= 4/9\, \Gamma(1/3)$

$\Gamma(4/3)$

$= \Gamma(1/3 + 1)$

$= 1/3\, \Gamma(1/3)$

(5) $L(e^t \sin^2 3t)$

$= L(\sin^2 3t)$

$= L\left(\frac{1-\cos 2\times 3t}{2}\right)$

$= L\left(\frac{1-\cos 6t}{2}\right)$

$= \frac{1}{2} L(1-\cos 6t)$

$= \frac{1}{2}\, L(1) - L(\cos 6t)$

$= \frac{1}{2} \times \left[\frac{1}{s} - \frac{s}{s^2+36}\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$

(6) $L\{\sin^2 t\}$

$= L\left\{\frac{1-\cos 2t}{2}\right\}$

$= \frac{1}{2} L\{1-\cos 2t\}$

$= \frac{1}{2}\, L\{1\} - L\{\cos 2t\}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

(7) $L\{t\cos at\}$

$= L\{\cos at\}$

$= \frac{s}{s^2+a^2}$

$L\{t\cos at\}$

$= -\frac{d}{ds}\left\{\frac{s}{s^2+a^2}\right\}$

$= -\frac{(s^2+a^2)\frac{d}{ds}s - s\frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2}$

$= -\frac{(s^2+a^2)\times 1 - s\times 2s}{(s^2+a^2)^2}$

$= -\frac{(s^2+a^2) - 2s^2}{(s^2+a^2)^2}$

$= \frac{-(s^2+a^2) + 2s^2}{(s^2+a^2)^2}$

(8) $L\left\{\frac{\sin 2t}{t}\right\}$

$= L\{\sin 2t\}$

$= \frac{2}{s^2+4}$

$L\left\{\frac{1}{t} \cdot \frac{2}{s^2+4}\right\}$

$= \int_s^\infty \frac{2}{s^2+4}\, ds$

$= \left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}\frac{s}{2}$

$= \frac{\pi}{4} - \tan^{-1}\frac{s}{2}$

$= \cot^{-1}\left(\frac{s}{2}\right)$

(9) $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

$= L^{-1}\left\{\frac{3s+7}{s^2-3s+s-3}\right\}$

$= L^{-1}\left\{\frac{3s+7}{s(s-3)+1(s-3)}\right\}$

$= L^{-1}\left\{\frac{3s+7}{(s-3)(s+1)}\right\}$

$\frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$

$= \frac{A(s+1)+B(s-3)}{(s-3)(s+1)}$

$= A(s+1)+B(s-3)$ ——— (1)

$s-3=0$
$s=3.$

$\Rightarrow 3.3+7 = A(3+1)+B(3-3)$

$\Rightarrow 9+7 = A(4)+B(0)$

$\Rightarrow 16 = 4A$

$\Rightarrow A = \cancel{16}/\cancel{4}\ 4$

$\Rightarrow A=4$

$s+1=0$
$s=-1$

$\Rightarrow 3\cdot(-1)+7 = A(-1+1)+B(-1-3)$

$\Rightarrow -3+7 = B(-4)$

$\Rightarrow 4 = -4B$

$\Rightarrow B = \cancel{-4}/4 = -1$

$\frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$

$= L^{-1}\left\{\frac{4}{s-3}\right\} + L^{-1}\left\{\frac{-1}{s+1}\right\}$

$= 4L^{-1}\left\{\frac{1}{s-3}\right\} - L^{-1}\left(\frac{1}{s+1}\right)$

$= 4\times e^{3t} - e^{-t}$

$= 4e^{3t} - e^{-t}$

(10) $L^{-1}\left\{\frac{2s}{s^2-9}\right\}$

$= 2L^{-1}\left\{\frac{s}{s^2-3^2}\right\}$

$= 2\times \cosh 3t$

$= 2\cosh 3t$

(11) $\cancel{\times} L^{-1}\left\{\frac{1}{(s+1)(s^2-1)}\right\}$

(13) $L\{f(t)\} = f(s)$

$L\{1\} = 1/s$

(12) $L\{t^3 e^{-3t}\}$

$= L\{t^3\}$

$= \frac{6}{s^4}$

$L\{t^3 e^{-3t}\}$

$= \frac{8}{(s+3)^4}$

(14) $L^{-1}\left\{\frac{4s-3}{s^2+9}\right\}$

$= L^{-1}\left\{\frac{4s}{s^2+9}\right\} - L^{-1}\left\{\frac{3}{s^2+9}\right\}$

$= 4L^{-1}\left\{\frac{s}{s^2+3^2}\right\} - L^{-1}\left\{\frac{3}{s^2+3^2}\right\}$

$= 4\times\cos 3t - \times \sin 3t$

$= 4\cos 3t - \sin 3t$

(15) $L(t \sin^2 t)$

$= L\{\sin^2 t\}$

$= L\left\{\frac{1-\cos 2t}{2}\right\}$

$= \frac{1}{2} L\{1-\cos 2t\}$

$= \frac{1}{2} L\{1\} - L\{\cos 2t\}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

(16) $L\{e^{2t}.t\}$

$= L\{t\}$

$= \frac{1}{s^2}$

$L\{e^{2t}.t\}$

$= \frac{1}{(s-2)^2}$

(17) $L\left\{\frac{\cos 2t - \cos 3t}{t}\right\}$

$= L\{\cos 2t - \cos 3t\}$

$= L\{\cos 2t\} - L\{\cos 3t\}$

$= \frac{s}{s^2+4} - \frac{s}{s^2+9}$

$L\left\{\frac{1}{t}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right)\right\}$

$= \int_s^\infty \frac{s}{s^2+4}\,ds - \int_s^\infty \frac{s}{s^2+9}\,ds$

$t = s^2+4$

$\frac{dt}{ds} = 2s$

$dt = 2s\,ds$

$\frac{1}{2} dt = s\,ds$

$v = s^2+9$

$\frac{dv}{ds} = 2s$

$dv = 2s\,ds$

$\frac{1}{2} dv = s\,ds$

$$= \frac{1}{2}\int_s^\infty \frac{dt}{t} - \frac{1}{2}\int_s^\infty \frac{dv}{v}$$

$$= \frac{1}{2}\left[\log t\right]_s^\infty - \frac{1}{2}\left[\log v\right]_s^\infty$$

$$= \cancel{\frac{1}{2}\left(\log\infty - \log\right.}$$

$$= \frac{1}{2}\left[\log(s^2+4)\right]_s^\infty - \frac{1}{2}\left[\log(s^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\infty - \log(s^2+4)\right] - \frac{1}{2}\left[\log\infty - \log(s^2+9)\right]$$

$$= \frac{1}{2}\left[-\ln(s^2+4)\right] - \frac{1}{2}\left[-\ln(s^2+9)\right]$$

$$= -\frac{1}{2}\ln(s^2+4) + \frac{1}{2}\ln(s^2+9)$$

$$= \frac{1}{2}\left[-\ln(s^2+4) + \ln(s^2+9)\right]$$

$$= \frac{1}{2}\left[\ln(s^2+9) - \ln(s^2+4)\right]$$

$$= \frac{1}{2}\left[\ln\frac{(s^2+9)}{(s^2+4)}\right]$$

(18)

$$L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

$$= \frac{A(s+2)^2 + B(s+2)}{(s+2)(s+2)^2}$$

(19) $L^{-1}\left\{\frac{2s-4}{(s-1)(s-2)(s-3)}\right\}$

$= \frac{2s-4}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$

$= \frac{A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)}{(s-1)(s-2)(s-3)}$

$= A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)$

$s-1=0$
$s=1$

$\Rightarrow 2\times1-4 = A(1-2)(1-3)$
$\Rightarrow 2-4 = A(-1)(-2)$
$\Rightarrow -2 = 2A$
$\Rightarrow A = -2/2 = -1.$

$s-2=0$
$s=2$

$\Rightarrow 2\cdot2-4 = B(2-1)(2-3)$
$\Rightarrow 4-4 = B(1)(-1)$
$\Rightarrow 0 = (-1)B$
$\Rightarrow B = 0.$

$s-3=0$
$s=3.$

$\Rightarrow 2\cdot3-4 = \cancel{A(3-2)}\, C(3-1)(3-2)$
$\Rightarrow 6-4 = C(2)(1)$
$\Rightarrow 2 = C(2)$
$\Rightarrow C = 2/2 = 1$

$\frac{2s-4}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$

$= \frac{-1}{s-1} + \frac{0}{s-2} + \frac{1}{s-3}$

$= \frac{-1}{s-1} + \frac{1}{s-3}$

$= L^{-1}\left\{\frac{-1}{s-1}\right\} + L^{-1}\left\{\frac{1}{s-3}\right\}$

$= -1\, L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{s-3}\right\}$

$= -1 \times e^{t} + e^{3t}$

$= -e^{t} + e^{3t}$

(20)

$\sin(A+B) = \sin A \cos B + \cos A \cdot \sin B$

$\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$

$\sin(A+B) + \sin(A-B) = 2 \sin A \cdot \sin B$

$\frac{1}{2} \sin(A+B) + \sin(A-B) = \sin A \cdot \sin B$

$L\{\sin 2t \cdot \cos 3t\}$

$= L\left\{\frac{1}{2}\sin(2t+3t) + \sin(2t-3t)\right\}$

$= \frac{1}{2} L\{\sin 5t + \sin(-t)\}$

$= \frac{1}{2} L\{\sin 5t + \sin(-t)\}$

$= \frac{1}{2} L\{\sin 5t - \sin t\}$

~~L{sin 5t~~

$L\{\sin 2t \cdot \cos 3t\}$

$= \frac{1}{2} L\{\sin 5t - \sin t\}$

$= \frac{1}{2} L\{\sin 5t\} - L\{\sin t\}$

$= \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]$

(21) $L\{(\sin t - \cos t)^2\}$

$= L\{\sin^2 t - 2\sin t \cdot \cos t + \cos^2 t\}$

$= L\{\sin^2 t + \cos^2 t - 2\sin t \cdot \cos t\}$

$= L\{1 - \sin 2t\}$

$= L\{1\} - L\{\sin 2t\}$

$= \frac{1}{s} - \frac{2}{s^2+4}$

$= \frac{s^2+4-2s}{s(s^2+4)} = \frac{s^2-2s+4}{s(s^2+4)}$

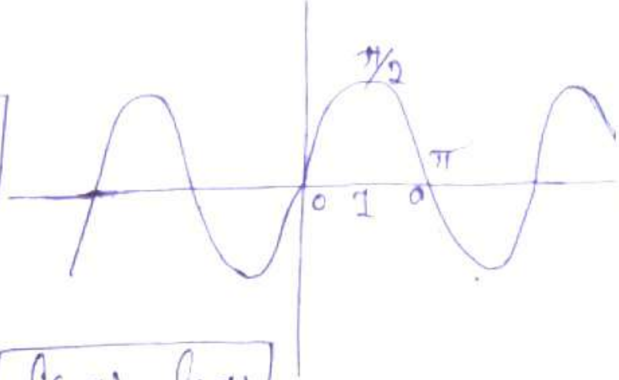
Fourier Series

* Periodic function :-

A function $f: R \longrightarrow R$

Then $\boxed{f(x+T) = f(x)}$

where 'T' is a constant.



* Even function -

If $f(x)$ is a function then $\boxed{f(-x) = f(x)}$ then the function is called even function.

e.g - $\cos x$; $\sec x$, x^2, x^4 .... etc...

The function which have even powers is called even function.

Ex-1.

let $f(x) = x^2$

$\Rightarrow f(-x) = (-x)^2 = x^2$

$\Rightarrow f(-x) = x^2$

* Odd function - If $f(x)$ is a function then $\boxed{f(-x) = -f(x)}$ then the function is called odd function.

e.g - $\sin x$, $\operatorname{cosec} x$, x, x^3, x^5 ....

Ex-1 let $f(x) = x^3$

$f(-x) = (-x)^3$

$f(-x) = -x^3$

$= -f(x)$

Standard formula

(1) $\int_{-a}^{a} f(x)\,dx = 0$, Here $f(x)$ is an odd function.

e.g... $\int_{-2}^{2} x^3\,dx = \left[\frac{x^4}{4}\right]_{-2}^{2} = \left[\frac{2^4}{4}\right] - \left[\frac{-2^4}{4}\right] = \frac{16}{4} - \frac{16}{4}$

$= 0$

(2) $\int_{-a}^{a} f(x) = 2\int_{0}^{a} f(x)$ here $f(x)$ is an even function.

$$= 2\int_{0}^{a} f(x)\,dx$$

e.g. $\int_{-2}^{2} x^2\,dx = \left[\frac{x^3}{3}\right]_{-2}^{2} = \frac{2^3}{3} - \frac{(-2)^3}{3} = \frac{8}{3} + \frac{8}{3}$

$$= \frac{16}{3}$$

$$2\int_{0}^{2} x^2\,dx = 2\times\left[\frac{x^3}{3}\right]_{0}^{2} = 2\times\left\{\frac{2^3}{3} - 0\right\}$$

$$= 2\left(\frac{8}{3}\right)$$

$$= \frac{16}{3}$$

(3) $\sin n\pi = \cos(n+\frac{1}{2})\pi = 0$

$\boxed{\text{For } \sin(n\pi) = 1, 2 \ldots n \quad \cos(n+\tfrac{1}{2})\pi = 0, 1 \ldots n}$

$\sin\pi = \cancel{\cos}\ \sin 2\pi = \sin 3\pi = 0.$

$\cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = \cos\frac{5\pi}{2} = 0$

(4) $\boxed{\cos(n\pi) = \sin(n+\frac{1}{2})\pi = (-1)^n}$

$\cos 0 = \cos 2\pi = \cos 4\pi \ldots\ldots 0$

$\cos\pi = \cos 3\pi = \cos 5\pi \ldots\ldots = -1$

$\sin\frac{\pi}{2} = \sin\frac{3\pi}{2} = \sin\frac{5\pi}{2} \ldots\ldots = 1$

Bernolli's Generalised Rule of Integration by parts:-

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$$

Here, u, u', u'', u''' .... etc.. are the successive derivative of u

$$u' = \frac{du}{dx}$$

$$v'' = \frac{d^2u}{dx^2}$$

$$v''' = \frac{d^3u}{dx^3} \ldots$$

v_1, v_2, v_3, v_4 .... etc. are the successive integration.

$$v_1 = \int v\,dx$$

$$v_2 = \int v_1\,dx$$

$$v_3 = \int v_2\,dx \ldots$$

Ex-1

$$\int x e^{3x}\,dx$$

$$= x\int e^{3x}\,dx - 1\int\left(\frac{e^{3x}}{3}\right)dx$$

$$= x \times \frac{e^{3x}}{3} - 1 \times \frac{1}{3}\left(\frac{e^{3x}}{3}\right)$$

$$= \frac{xe^{3x}}{3} - \frac{1}{3}\left(\frac{e^{3x}}{3}\right)$$

$$= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}$$

Ex-1

$$\int (x+x^2)\cos nx\,dx$$

$$= (x+x^2)\int \cos nx\,dx - (1+2x)\int \frac{\sin nx}{n}$$

$$= (x+x^2)\frac{\sin nx}{n} - (1+2x) \times \frac{1}{n}\int \sin nx$$

$$= (x+x^2)\frac{\sin nx}{n} - (1+2x) \times \frac{1}{n}\,\frac{-\cos nx}{n}$$

$$= (x+x^2)\frac{\sin nx}{n} - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + 2\int \frac{-\cos nx}{n^2}$$

$\int \cos nx = \frac{\sin nx}{n}$

$\int \cos 2x = \frac{\sin 2x}{2}$

$\int \cos 3x = \frac{\sin 3x}{3}$

$$= (x+x^2)\frac{\sin nx}{n} + (1+2x)\frac{\cos nx}{n^2} + 2\times \frac{1}{n^2}\int -\cos nx$$

$$= (x+x^2)\frac{\sin nx}{n} + (1+2x)\frac{\cos nx}{n^2} + 2\times \frac{1}{n^2}\left(\frac{-\sin nx}{n}\right)$$

$$= (x+x^2)\left(\frac{\sin(nx)}{n}\right) + (1+2x)\left(\frac{\cos nx}{n^2}\right) - 2\left(\frac{\sin nx}{n^3}\right)$$

$$* \int e^{ax} \sin bx \, dx$$

$$\cancel{= \sin bx \int e^{ax} dx - b\cos bx}$$

$$= \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right] + C$$

$$* \int e^{ax} \cos bx \, dx$$

$$= \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + C$$

Fourier Series

A series of from $f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Euler's thm / Formula

The fourier series of the fun $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (i)}$$

Hence, $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\,dx$$

Working Rule :- Let $f(x)$ be a function with interval $-\pi < x < \pi$.

Step-1

First to check the function is even or odd.

$f(x) = x, \quad -\pi < x < \pi$

Step-2

(a) If the function $f(x)$ is an even, then, $b_n = 0$.

(b) If the function $f(x)$ is an odd, then $a_n = 0$

Step-3

If a function is neither even nor odd, solving by Euler's formula.

Imp
Ex-1

Find the Fourier expansion of $f(x) = x$ in $-\pi < x < \pi$

Sol^n Let $f(x) = x$

Since $f(x)$ is an odd function so $a_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx \quad \underline{\underline{\text{or}}} = \frac{2}{\pi}\int_{0}^{2\pi} x\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} \qquad\qquad = \frac{1}{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{(-\pi)^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi} \times 0$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin nx \, dx$$

$$= \frac{1}{\pi}\left\{x\int \sin nx\,dx - 1\int \frac{-\cos nx}{n}\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{x \times \left(\frac{-\cos nx}{n}\right) - 1 \times \frac{1}{n} \frac{-\sin nx}{n}\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n^2} - \frac{x\cos nx}{n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi}{n^2} - \frac{\pi\cos n\pi}{n}\right] - \left[\frac{\sin(-n\pi)}{n^2} - \frac{-\pi\cos(-n\pi)}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi}{n^2} - \frac{\pi\cos n\pi}{n}\right] - \left[\frac{-\sin n\pi}{n^2} + \frac{\pi\cos n\pi}{n}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\sin n\pi}{n^2}}^{0} - \frac{\pi\cos n\pi}{n}\right\} + \cancel{\frac{\sin n\pi}{n^2}}^{0} - \frac{\pi\cos n\pi}{n}\Big]$$

$$= \frac{1}{\cancel{\pi}}\left[-\frac{2\cancel{\pi}\cos n\pi}{n}\right]$$

$$= -2\left(\frac{\cos n\pi}{n}\right)$$

Hence, $f(x) = \sum_{n=1}^{\infty} -2\left(\frac{\cos n\pi}{n}\right)\sin nx$

$$= -2\left(\frac{\cos\pi \cdot \sin\pi}{1}\right) + -2\left(\frac{\cos 2\pi \cdot \sin 2\pi}{2}\right) + -2\left(\frac{\cos 3\pi \cdot \sin 3\pi}{3}\right) + \dots$$

Ex-1 Find the Fourier Series of $f(x) = x - x^2$ in $-\pi < x < \pi$

Let $f(x) = x - x^2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---} (i)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,dx - \int_{-\pi}^{\pi} x^2\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} - \frac{-\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[-\frac{2\cancel{\pi^3}}{3}\right]$$

$$= \frac{-2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x-x^2)\left(\frac{\sin nx}{n}\right) - (1-2x)\left(\frac{1}{n}\frac{-\cos nx}{n}\right) + (-2)\left(\frac{1}{n^2}\frac{-\sin nx}{n}\right)$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{\sin nx}{n}\right) - (1-2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\cancel{(x-x^2)\left(\frac{\sin nx}{n}\right)}^{0} + (1-2x)\left(\frac{\cos nx}{n^2}\right) + \cancel{2\left(\frac{\sin nx}{n^3}\right)}^{0}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1-2x)\left(\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi} \qquad \{\because \sin n\pi = 0\}$$

$$= \frac{1}{\pi}\left[(1-2\pi)\left(\frac{\cos n\pi}{n^2}\right) - (1+2\pi)\left(\frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos n\pi}{n^2}\right)(\cancel{1} - 2\pi - \cancel{1} - 2\pi)\right]$$

$$= \frac{1}{\cancel{\pi}}\left[-4\cancel{\pi}\left(\frac{\cos n\pi}{n^2}\right)\right]$$

$$= -4\left(\frac{\cos n\pi}{n^2}\right)$$

$$= -4\left(\frac{(-1)^n}{n^2}\right) \qquad \{\because \cos n\pi = (-1)^n\}$$

$$= \frac{-4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right) - (1-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right) + \cancel{(1-2x)\left(\frac{\sin nx}{n^2}\right)} - 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(x-x^2)\left(\frac{\cos nx}{n}\right) - 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi} \qquad \{\sin n\pi = 0\}$$

$$= -\frac{1}{\pi}\left[(x-x^2)\left(\frac{\cos nx}{n}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{\pi}\left[(\pi-\pi^2)\left(\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right)\right] - \left[(-\pi-\pi^2)\left(\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right)\right]$$

$$= -\frac{1}{\pi}\left[(\pi-\pi^2)\left(\frac{\cos n\pi}{n}\right) + \cancel{2\left(\frac{\cos n\pi}{n^3}\right)} - (-\pi-\pi^2)\left(\frac{\cos n\pi}{n}\right) - \cancel{2\left(\frac{\cos n\pi}{n^3}\right)}\right]$$

$$= -\frac{1}{\pi}\left[(\pi-\pi^2)\left(\frac{\cos n\pi}{n}\right) - (-\pi-\pi^2)\left(\frac{\cos n\pi}{n}\right)\right]$$

$$= -\frac{1}{\pi}\left[(\pi-\pi^2)\left(\frac{\cos n\pi}{n}\right) - [-(\pi+\pi^2)]\left(\frac{\cos n\pi}{n}\right)\right]$$

$$= -\frac{1}{\pi}\left[(\pi-\pi^2)\left(\frac{\cos n\pi}{n}\right) + (\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right)\right]$$

$$= -\frac{1}{\pi}\left(\frac{\cos n\pi}{n}\right)\left[\pi - \cancel{\pi^2} + \pi + \cancel{\pi^2}\right]$$

$$= -\frac{1}{\cancel{\pi}}\left(\frac{\cos n\pi}{n}\right) 2\cancel{\pi} \qquad = \frac{-2}{n}(\cos n\pi) = -\frac{2}{n}(-1)^n$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \cancel{\frac{-2\pi^2}{3 \cdot 2}} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\cancel{\frac{-2\pi^2}{6}}$$

$$= \frac{-\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{-4(-1)^n}{n^2}\right) \cos nx + \sum_{n=1}^{\infty} \left(\frac{-2(-1)^n}{n}\right) \sin nx$$

$$= \frac{-\pi^2}{3} + (-4) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx + (-2) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

$$\cancel{= \frac{-\pi^2}{3} + (-4)\left(\frac{-1}{1}\cos x\right) + (-2)\Big(}$$

$$= \frac{-\pi^2}{3} + (-4)\left(\frac{-\cos x}{1} + \frac{\cos 2x}{4} - \frac{\cos 3x}{9}\right) + \cdots + (-2)\left(\frac{-\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3}\right) \cdots$$

Read Fourier Series.

$$x - x^2 = -\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Q- Find a series of sines and cosines of multiple of x which will represent $x+x^2$ in the interval $-\pi < x < \pi$.

Let, $f(x) = x+x^2$

Then the fourier series of $f(x) = x+x^2$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---} (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$= \left[\frac{2\pi^2}{3}\right].$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{(x+x^2)}^{u}\overbrace{\cos nx}^{v}\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{\sin nx}{n}\right) - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x+x^2)\cancel{\left(\frac{\sin nx}{n}\right)}^{0} + (1+2x)\left(\frac{\cos nx}{n^2}\right) - 2\cancel{\left(\frac{\sin nx}{n^3}\right)}^{0}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2x)\left(\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2\pi)\left(\frac{\cos n\pi}{n^2}\right) - (1-2\pi)\left(\frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos n\pi}{n^2}\right)\left(\cancel{1}+2\pi-\cancel{1}+2\pi\right)\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{\cos n\pi}{n^2}\times 4\cancel{\pi}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right)-(1+2x)\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(x+x^2)\left(\frac{\cos nx}{n}\right)+(1+2x)\cancel{\left(\frac{\sin nx}{n^2}\right)}^{0}+2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(x+x^2)\left(\frac{\cos nx}{n}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[-(\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right)+2\left(\frac{\cos n\pi}{n^3}\right)\right]-\left[-(-\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right)+2\left(\frac{\cos n\pi}{n^3}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right)+\cancel{2\left(\frac{\cos n\pi}{n^3}\right)}+(-\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right)-\cancel{2\left(\frac{\cos n\pi}{n^3}\right)}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n}\left(-\pi-\cancel{\pi^2}-\pi+\cancel{\pi^2}\right)\right]$$

$$= \frac{1}{\cancel{\pi}}\left(\frac{\cos n\pi}{n}(-2\cancel{\pi})\right)$$

$$= \frac{-2\cos n\pi}{n}$$

$$= \frac{-2\times(-1)^n}{n}$$

$$\therefore f(x) = \frac{\left(\frac{\cancel{2}\pi^2}{3}\right)}{\cancel{2}} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n^2}\sin nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx + (-2)\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\sin nx$$

$$= \frac{\pi^2}{3} + 4\left(-\cos x + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + \cdots\right) + (-2)\left(-\sin x + \frac{\sin 2x}{4} - \frac{\sin 3x}{9}\right) \quad (2)$$

Prove $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Put, $x = \pi$

$$= \frac{\pi^2}{3} + 4\left(-\cos\pi + \frac{\cos 2\pi}{4} - \frac{\cos 3\pi}{9} + \cdots\right)$$

$$= \frac{\pi^2}{3} + 4\left(1 + \frac{1}{4} + \frac{1}{3} + \cdots\right)$$

$$\Rightarrow x + x^2 = \frac{\pi^2}{3} + 4\left(1 + \frac{1}{4} + \frac{1}{3} + \cdots\right)$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left(1 + \frac{1}{4} + \frac{1}{3} + \cdots\right)$$

$$\Rightarrow \frac{2\pi^2}{4\times 3} = \left(1 + \frac{1}{4} + \frac{1}{3} + \cdots\right)$$

$$\Rightarrow \frac{\cancel{2}\pi^2}{\cancel{12}\,6} = \left(1 + \frac{1}{4} + \frac{1}{3^2} + \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{6} = \left(1 + \frac{1}{4} + \frac{1}{3^2} + \cdots\right).$$

Q) Find the fourier series of e^x in the interval $= -\pi < x < \pi$.

$\frac{e^x - e^{-x}}{2} = \sinh x$

$\frac{e^x + e^{-x}}{2} = \cosh x$

let, $f(x) = e^x$

Then the fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---(1)}$$

then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \, dx$

$= \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$

$= \frac{1}{\pi} \times 2\sinh \pi$

$= \frac{2\sinh \pi}{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx \, dx \quad (a=1, b=n)$

$= \frac{1}{\pi}\left[\frac{e^x}{1^2+n^2}\left\{\cos nx + n\sin nx\right\}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi(1+n^2)}\left[e^x \cos nx\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi(1+n^2)}\left[e^x \cos nx\right]_{-\pi}^{\pi} \quad \{\because \sin n\pi = 0\}$

$= \frac{1}{\pi(1+n^2)}\left\{e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right\}$

$= \frac{\cos n\pi}{\pi(1+n^2)}\left\{e^{\pi} - e^{-\pi}\right\} = \frac{(-1)^n \times 2\sinh\pi}{\pi(1+n^2)}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx \, dx \quad (a=1, b=n)$

$= \frac{1}{\pi}\left[\frac{e^x}{1^2+n^2}\left\{\sin nx - n\cos nx\right\}\right]$

$= \frac{1}{\pi(1+n^2)}\left[-e^x n\cos nx\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi(1+n^2)}\left[-e^{\pi} n\cos n\pi + e^{-\pi} n\cos n\pi\right]$

$= \frac{n\cos n\pi}{\pi(1+n^2)}\left[-e^{\pi} + e^{-\pi}\right]$

$= -\frac{n\cos n\pi}{\pi(1+n^2)}\left(e^{\pi} - e^{-\pi}\right)$

$= \frac{-n\cos n\pi \, 2\sinh\pi}{\pi(1+n^2)}$

$= \frac{-n(-1)^n 2\sinh\pi}{\pi(1+n^2)}$

$$f(x) = \frac{\frac{2\sinh\pi}{\pi}}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \times 2\sinh\pi}{\pi(1+n^2)} \cos nx + \sum_{n=1}^{\infty} \frac{-n(-1)^n \sinh\pi}{\pi(1+n^2)} \sin nx$$

Dirichlet condition

Any function $f(x)$ can be expressed as a fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\, dx + \sum_{n=1}^{\infty} b_n \sin nx\, dx$ in interval $[\alpha, \alpha+2\pi]$ where a_0, a_n, b_n are constants.

(i) $f(x)$ is periodic, single valued & finite

(ii) $f(x)$ has finite number of finite discontinuities

(iii) $f(x)$ has only finite number of local maxima & Minima.

fourier Series of Discontinious function

let, the function $f(x)$ defined by $f(x) = \begin{cases} f_1(x), & \alpha < x < x_0 \\ f_2(x), & x_0 < x < \alpha+2\pi \end{cases}$

$$a_0 = \cancel{\frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi}} \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\cos nx\,dx\right]$$

Q) $f(x)$, if $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

The fourier Series of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\, dx + \int_{0}^{\pi} x\, dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\, dx + \int_{0}^{\pi} x \cos nx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} +\right.$$

$$= \frac{1}{\pi}\left[-\pi \int_{-\pi}^{0} dx + \left(\frac{x^2}{2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi\,[x]_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-\pi)(0+\pi) + \frac{\pi^2}{2} - 0\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{1} + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi^2 + \pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{2}\right]$$

$$= \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n^2}\right)_0^{\pi}\right] \qquad \boxed{\because \sin n\pi = 0}$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos n\pi}{n^2}\right) - \frac{\cos n0}{n^2}\right]$$

$$= \frac{1}{\pi}\left(\frac{\cos n\pi - \cos n0}{n^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{\cos n\pi - 1}{n^2}\right)$$

$$= \frac{1}{\pi}\,\frac{(-1)^n - 1}{n^2}$$

$$= \frac{(-1)^n - 1}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx \, dx + \int_{0}^{\pi} x \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left[x \times \frac{-\cos nx}{n} - 1 \times \frac{-\sin nx}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{-\cos 0}{n} + \frac{\cos n(-\pi)}{n}\right) + \left[x\frac{-\cos nx}{n} + \cancel{\frac{\sin nx}{n^2}}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[+\pi\left(\frac{1}{n} - \frac{\cos n\pi}{n}\right)\right] + \left[\pi\left(\frac{-\cos n\pi}{n}\right) + 0\left(\frac{\cos 0}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{1}{n} - \frac{\cos n\pi}{n}\right) - \frac{\pi \cos n\pi}{n}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\cancel{\pi}\left(\frac{1}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n}\right)\right.$$

$$= \frac{1}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n}$$

$$= \frac{1 - 2\cos n\pi}{n}$$

$$= \frac{1 - 2(-1)^n}{n}$$

1) Find the period of $\sin 3x$

$= \frac{2\pi}{3}$

2) Find a_0 for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx$

$= \frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left(-e^{-2\pi} + e^{0}\right)$

$= \frac{1}{\pi}\left(-e^{-2\pi} + 1\right)$

$= \frac{-e^{-2\pi} + 1}{\pi}$

3) Find the fourier co-efficient a_0 and a_n for $f(x)$

$\begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_0^{\pi} x\,dx\right]$

$= \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0} 1\,dx + \int_0^{\pi} x\,dx\right]$

$= \frac{1}{\pi}\left[-\pi\left(x\right)_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right]$

$= \frac{1}{\pi}\left[-\pi(0+\pi) + \left(\frac{\pi^2}{2} - 0\right)\right]$

$= \frac{1}{\pi}\left[-\pi(\pi) + \frac{\pi^2}{2}\right]$

$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right]$

$= \frac{1}{\pi}\left[\frac{-2\pi^2 + \pi^2}{2}\right]$

$= \frac{1}{\pi}\left(\frac{-\pi^2}{2}\right)$

$= \frac{-\pi}{2}$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\,dx + \int_{0}^{\pi} x \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0} \cos nx\,dx + \int_{0}^{\pi} x \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin n(0)}{n} - \cancel{\frac{\sin n(-\pi)}{n}}^{0}\right) + \left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\sin n\pi}{n}\right) - 1\left(\frac{-\cos n\pi}{n^2}\right) - 0\left(\frac{\sin n0}{n}\right) - 1\left(\frac{\cos n0}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\cancel{\frac{\sin n\pi}{n}}^{0}\right) + \frac{\cos n\pi}{n^2} - 1\frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right].$$

$$= \frac{1}{\pi}\left(\frac{\cos n\pi - 1}{n^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$= \frac{(-1)^n - 1}{\pi n^2}$$

(4) find the Fouriere coefficient fore $f(x) = x^3, -\pi < x < \pi$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^4}{4}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^4}{4} - \frac{-\pi^4}{4}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^4}{4}} - \cancel{\frac{\pi^4}{4}}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos nx \, dx$$

$$= \frac{1}{\pi}\left[x^3 \times \frac{\sin nx}{n} - \frac{x^2}{2} \times \frac{-\cos nx}{n^2} + \frac{x}{2} \times \frac{-\sin nx}{n^3} - \frac{1}{2}\frac{\cos nx}{n^4}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\cancel{\frac{x^3 \sin nx}{n}} + \frac{x^2 \cos nx}{2n^2} - \cancel{\frac{x}{2}\frac{\sin nx}{n^3}} - \frac{1}{2}\frac{\cos nx}{n^4}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2 \cos nx}{2n^2} - \frac{\cos nx}{2n^4}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2 \cos n\pi}{2n^2} - \frac{\cos n(\pi)}{2n^4}\right) - \left(\frac{-\pi^2 \cos n(-\pi)}{2n^2} - \frac{\cos n(-\pi)}{2n^4}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2 \cos n\pi}{2n^2} - \frac{\cos n\pi}{2n^4}\right) + \left(\frac{\pi^2 \cos n\pi}{2n^2} + \frac{\cos n\pi}{2n^4}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2 \cos n\pi}{2n^2} - \cancel{\frac{\cos n\pi}{2n^4}} + \frac{\pi^2 \cos n\pi}{2n^2} + \cancel{\frac{\cos n\pi}{2n^4}}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2 \cos n\pi}{2n^2} + \frac{\pi^2 \cos n\pi}{2n^2}\right]$$

$$= \frac{1}{\pi}\left(\frac{\cancel{2}\pi^2 \cos n\pi}{\cancel{2}n^2}\right)$$

$$= \frac{1}{\pi} \times \frac{\pi^2 \cos n\pi}{n^2}$$

$$= \frac{\pi \cos n\pi}{n^2}$$

$$= \frac{\pi(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x^3 \times \frac{-\cos nx}{n} - \frac{x^2}{2} \times \frac{-\sin nx}{n^2} + \frac{x}{2} \times \frac{\cos nx}{n^3} - \frac{1}{2}\times\frac{\sin nx}{n^4}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[x^3 \times \frac{-\cos nx}{n} + \cancel{\frac{x^2 \sin nx}{2n^2}} + \frac{x\cos nx}{2n^3} - \cancel{\frac{\sin nx}{2n^4}}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x^3\cos nx}{n} + \frac{x\cos nx}{2n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi^3\cos n\pi}{n} + \frac{\pi\cos n\pi}{2n^3}\right) - \left(\frac{-\pi^3\cos n(-\pi)}{n} + \frac{-\pi\cos n(-\pi)}{2n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{-\pi^3\cos n\pi}{n}} + \frac{\pi\cos n\pi}{2n^3} + \cancel{\frac{\pi^3\cos n\pi}{n}} - \frac{-\pi\cos n\pi}{2n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\cos n\pi}{2n^3} - \frac{-\pi\cos n\pi}{2n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\cos n\pi}{2n^3} + \frac{\pi\cos n\pi}{2n^3}\right]$$

$$= \frac{1}{\pi}\left(\frac{\cancel{2}\cos n\pi}{\cancel{2}n^3}\right)$$

$$= \frac{1}{\pi}\frac{\cos n\pi}{n^3}$$

$$= \frac{(-1)^n}{\pi n^3}$$

(6) $\Gamma(1/2)$

$\cancel{-1/2\,\Gamma\,-1/2} = \sqrt{\pi}$

$= \cancel{\frac{1}{\Gamma(1/2+1)}}$

$= \cancel{2\sqrt{\Gamma(3/2)}}$

$\frac{1}{2} - \frac{1}{1}$

$= \frac{1-2}{2}$

$= -1/2$

$= -1/2 - 1$

$= \frac{1-2}{2}$

$= 3/2$

(7) Find the value of fourier co-efficient a_0 if $f(x) = x + x^2 (-\pi, \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x + x^2 \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \, dx + \int_{-\pi}^{\pi} x^2 \, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} + \left[\frac{x^3}{3}\right]_{-\pi}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\cancel{\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right)} + \left(\frac{\pi^3}{3} - \frac{-\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[+\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\cancel{\pi}}\left(\frac{2\pi^3}{3}\right)$$

$$= \frac{2\pi^2}{3}.$$

(8) a

(9) obtain the Fourier series for $f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\int_{-\pi}^{0} dx + k\int_{0}^{\pi} dx\right]$$

$$= \frac{1}{\pi}\left[-k\,[x]_{-\pi}^{0} + k\,[x]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-k\,(0+\pi) + k\,(\pi-0)\right]$$

$$= \frac{1}{\pi}\left[-k(\pi) + k\pi\right]$$

$$= \frac{1}{\pi}\left[\pi(-k+k)\right].$$

$$= 0.$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\int_{-\pi}^{0}\sin nx\,dx + k\int_{0}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} + k\left(-\frac{\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-k\times\left(-\frac{\cos 0}{n} + \frac{\cos n(-\pi)}{n}\right) + k\times\left(\frac{-\cos n\pi}{n} + \frac{\cos 0}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[-k\left(\frac{1}{n} + \frac{\cos n\pi}{n}\right) + k\left(\frac{-\cos n\pi}{n} + \frac{1}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[-k\left(\frac{1+\cos n\pi}{n}\right) + k\left(\frac{-\cos n\pi + 1}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{-k}{n}} + \frac{-k\cos n\pi}{n} + \cancel{\frac{k}{n}} - \frac{k\cos n\pi}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2k\cos n\pi}{n}\right]$$

$$= \frac{-2K}{\pi n} \cos n\pi$$

$$= \frac{-2K(-1)^n}{\pi n}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -K \cos nx\,dx + \int_{0}^{\pi} K \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-K\int_{-\pi}^{0} \cos nx\,dx + K\int_{0}^{\pi} \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-K\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + K\left(\frac{\sin nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-K\left(\frac{\sin n0}{n} - \frac{\sin n(-\pi)}{n}\right) + K\left(\cancel{\frac{\sin n\pi}{n}}^{0} - \cancel{\frac{\sin n0}{n}}\right)\right]$$

$$= \frac{1}{\pi}\left[-K\left(\frac{\sin n0}{n} - \frac{-\sin n\pi}{n}\right) + K \times 0\right]$$

$$= \frac{1}{\pi}\left[-K\left(0 + \cancel{\frac{\sin n\pi}{n}}^{0}\right) + 0\right]$$

$$= \frac{1}{\pi} \times 0$$

$$= 0$$

(10) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$

$\cancel{a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx}$ $\quad a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x^2\,dx\right]$

$\cancel{= 0 =}$ $\quad = \frac{1}{\pi}\left[0 + \left(\frac{x^3}{3}\right)_0^{\pi}\right]$

$\cancel{a_n = \frac{1}{\pi}}$

$= \frac{1}{\pi}\left[0 + \frac{\pi^3}{3} - 0\right]$

$= \frac{1}{\pi}\left[\frac{\pi^3}{3}\right]$

$= \frac{\pi^2}{3}$

$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \sin nx\,dx + \int_{0}^{\pi} x^2 \sin nx\,dx\right]$

$= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} x^2 \sin nx\,dx\right]$

$= \frac{1}{\pi}\left[0 + \left(x^2 \times \frac{-\cos nx}{n} - 2x \times \frac{-\sin nx}{n^2} + 2 \times \frac{\cos nx}{n^3}\right)\right]_0^{\pi}$

$= \frac{1}{\pi}\left[\cancel{0+} -\frac{x^2 \cos nx}{n} + \frac{2x \cancelto{0}{\sin nx}}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{\pi}$

$= \frac{1}{\pi}\left[\frac{-x^2 \cos nx}{n} + \frac{2\cos nx}{n^3}\right]_0^{\pi}$

$= \frac{1}{\pi}\left[\left(\frac{-\pi^2 \cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right) - \left(0 + \frac{2\cos n0}{n^3}\right)\right]$

$= \frac{1}{\pi}\left[\frac{-\pi^2 \cos n\pi}{n} + \frac{2\cos n\pi}{n^3} - \frac{2}{n^3}\right]$

$= \frac{1}{\pi}\left[\frac{-\pi^2 n^2 \cos n\pi + 2\cos n\pi - 2}{n^3}\right]$

$$= \frac{1}{\pi}\left[\cos n\pi\,(-\pi^2 n^2 + 2 - 2)\right.$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2 n^2 (-1)^n + 2(-1)^n - 2}{n^3}\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cos nx\, dx + \int_{0}^{\pi} x^2 \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} x^2 \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[x^2 \times \frac{\sin nx}{n} - 2x \times \frac{-\cos nx}{n^2} + 2\,\frac{-\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\cancel{\frac{x^2 \sin nx}{n}}^{\,0} + \frac{2x\cos nx}{n^2} - \cancel{\frac{2\sin nx}{n^3}}^{\,0}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{2\pi\cos n\pi}{n^2} - \cancel{\frac{2\times 0 \cos n0}{n^2}}\right)$$

$$= \frac{1}{\pi}\left(\frac{2\pi\cos n\pi}{n^2}\right)$$

$$= \frac{2\cos n\pi}{n^2}$$

$$= \frac{2(-1)^n}{n^2}$$

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \frac{\pi x}{4}, & 0 < x < \pi \end{cases}$$

Let, $f(x)$ be represented by fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——— (1)}$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} f_1(x)\,dx + \int_{0}^{\pi} f_2(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \frac{\pi x}{4}\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi}{4}\int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4}\left[\frac{x^2}{2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4}\left(\frac{\pi^2}{2} - 0\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi^3}{8}\right) = \frac{\pi^2}{8}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f_1(x)\cos nx\,dx + \int_{0}^{\pi} f_2(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,dx + \int_{0}^{\pi} \frac{\pi x}{4}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi}{4}\int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4}\int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{4}\left[x \times \cancel{\frac{\sin nx}{n}}^{\,0} - 1 \times \frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] = \frac{1}{4}\left[\frac{\cos n\pi}{n^2} - \frac{1}{1}\right] = \frac{1}{4}\left[\frac{\cos n\pi - 1}{n^2}\right] = \frac{(-1)^n - 1}{4n^2}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f_1(x)\sin nx\,dx + \int_{0}^{\pi} f_2(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,\sin nx\,dx + \int_{0}^{\pi} \frac{\pi x}{4}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi}{4}\int_{0}^{\pi} x\sin nx\,dx\right]_0^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{4}\left(\int_{0}^{\pi} x\sin nx\,dx\right)\right]_0^{\pi}$$

$$= \frac{1}{4}\left(x\,\frac{-\cos nx}{n} - 1\times\frac{-\sin nx}{n^2}\right)_0^{\pi}$$

$$= \frac{1}{4}\left(-\frac{x\cos nx}{n} + \cancel{\frac{\sin nx}{n^2}}\right)_0^{\pi}$$

$$= \frac{1}{4}\left(-\frac{x\cos nx}{n}\right)_0^{\pi}$$

$$= \frac{1}{4}\left(-\frac{\pi\cos n\pi}{n} - \cancel{-\frac{0\cos n 0}{n}}\right)$$

$$= \frac{1}{4}\left(\frac{-\pi\cos n\pi}{n}\right)$$

$$= \frac{-\pi\cos n\pi}{4n} = \frac{-\pi(-1)^n}{4n}$$

Then fourier Series :- $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$

$$= \frac{\frac{\pi^2}{8}}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n-1}{4n^2}\cos nx + \sum_{n=1}^{\infty}\frac{-\pi(-1)^n}{4n}\sin nx$$

$$= \frac{\pi^2}{16} + \left(\frac{-2}{4}\cos x - \frac{2}{36}\cos 3x \ldots\right) + \frac{\pi}{4}\sin x - \frac{\pi}{8}\sin 2x + \frac{\pi}{12}\sin 3x -$$

(11) Define periodic function

A function (f) $R \longrightarrow R$

Then $f(x+T) = f(x)$

where 'T' is a constant.

ex :-

(12) Dirichlet's condition in Fourier Series -

Any function $f(x)$ can be expressed as a Fourier Series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, dx + \sum_{n=1}^{\infty} b_n \sin nx \, dx$ is interval $[\alpha, \alpha+2\pi]$ where $\boxed{n=1}$, a_0, a_n, b_n are constants.

(i) $f(x)$ is periodic, single valued & finite

(ii) $f(x)$ has finite number of finite discontinuities.

(iii) $f(x)$ has only finite number of local maxima & Minima.

$$f(x) = 2 \text{ in } 0 \leq x \leq 2\pi$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} 2 \, dx$$

$$= \frac{1}{\pi} 2 \int_0^{\pi} 1 \, dx$$

$$= \frac{1}{\pi} 2 \, [x]_0^{\pi}$$

$$= \frac{1}{\pi} 2 \, (\pi - 0)$$

$$= \frac{1}{\pi} 2 \, (\pi)$$

$$= \frac{1}{\pi} 2\pi = 2$$

(13)

(14) $f(x) = x$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right)$$

$$= 0.$$

(15) fun $f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \end{cases}$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} 0\,dx + \int_{\pi}^{2\pi} 1\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + (x)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[x\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left(2\pi - \pi\right)$$

$$= \frac{1}{\cancel{\pi}}\cancel{\pi}$$

$$= 1$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} 0 \sin nx\,dx + \int_{\pi}^{2\pi} 1 \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_{\pi}^{2\pi} \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos n(2\pi)}{n} + \frac{\cos n\pi}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi - \cos(2n\pi)}{n}\right]$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} \pi \cos nx \, dx + \int_{\pi}^{2\pi} 1 \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\pi \int_0^{\pi} \cos nx \, dx + 1 \int_{\pi}^{2\pi} \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\pi \times \cancel{\left(\frac{\sin nx}{n}\right)_0^{\pi}}^{0} + 1 \times \left(\frac{\sin nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\sin n 2\pi}{n} - \frac{\sin n\pi}{n}}^{0}\right]$$

$$= 0.$$

(16) $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \frac{\pi x}{4}, & 0 < x < \pi. \end{cases}$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} \frac{\pi x}{4}\, dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi}{4}\int_0^{\pi} x\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4} \times \left(\frac{x^2}{2}\right)_0^{\pi}\right]$$

$$= \frac{1}{4}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{\pi^2}{2} - 0\right]$$

$$= \frac{1}{4} \times \frac{\pi^2}{2}$$

$$= \frac{\pi^2}{8}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cos nx\, dx + \int_0^{\pi} \frac{\pi x}{4} \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \frac{\pi}{4}\int_0^{\pi} x \cos nx\, dx\right]$$

$$\cancel{= \frac{1}{\pi}\left[0 + \frac{\pi}{4}\left(\frac{\sin nx}{n}\right)_0^{\pi}\right]}$$

$$\cancel{= \frac{1}{\pi}\left[\frac{\pi}{4}\left(\frac{\sin nx}{n}\right)_0^{\pi}\right]}$$

$$\cancel{= \frac{1}{4}\left(\frac{\sin nx}{n}\right)_0^{\pi}}$$

$$\cancel{= 0}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4}\left(x \frac{\sin nx}{n} - 1 \times \frac{-\cos nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4}\left(\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{\cos n\pi}{n^2} - \frac{\cos n0}{n^2}\right] = \frac{1}{4}\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right)$$

$$= \frac{1}{4}\left(\frac{\cos n\pi - 1}{n^2}\right) = \frac{(-1)^n - 1}{4n^2}.$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \sin nx\, dx + \int_{0}^{\pi} \frac{\pi x}{4} \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi}{4}\int_{0}^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4} \left(x \times \frac{-\cos nx}{n} - 1 \times \frac{-\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{4} \times \left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\left(\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right) - \left(\frac{-0\cos n0}{n} + \frac{\sin n0}{n^2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{-\pi\cos n\pi}{n} + \cancelto{0}{\frac{\sin n\pi}{n^2}} + 0 - \cancelto{0}{\frac{\sin n0}{n^2}}\right]$$

$$= \frac{1}{4}\left(\frac{-\pi\cos n\pi}{n}\right)$$

$$= \frac{-\pi(-1)^n}{4n}$$

Fourier Series:—

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\frac{\pi^2}{8}}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{4n^2} \cos nx + \sum_{n=1}^{\infty} \frac{-\pi(-1)^n}{n} \sin nx$$

$$= \frac{\pi^2}{16} + \left(\frac{-2}{4}\cos nx - \frac{2}{36}\cos 3x\right) + \frac{\pi}{4}\sin nx - \frac{\pi}{8}\sin 2x$$

$$\frac{\pi}{12}\sin 3x -$$

(17) $f(x) = x^2, \quad -\pi < x \le \pi$

$\sum_{n=1}^{\infty} 1/n^2 = \frac{\pi^2}{6}$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$

$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left(\frac{\pi^3}{3} - \frac{-\pi^3}{3}\right)$

$= \frac{1}{\pi}\left(\frac{\pi^3}{3} + \frac{\pi^3}{3}\right)$

$= \frac{1}{\pi}\left(\frac{\pi^3+\pi^3}{3}\right)$

$= \frac{1}{\pi} \times \frac{2\pi^3}{3}$

$= \frac{2\pi^2}{3}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx$

$= \frac{1}{\pi}\left[x^2 \times \frac{-\cos nx}{n} - 2x \times \frac{-\sin nx}{n^2} + 2 \times \frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \cancel{\frac{2x\sin nx}{n^2}}^{0} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\left(\frac{-\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right) - \left(\frac{-\pi^2\cos n(-\pi)}{n} + \frac{2\cos n(-\pi)}{n^3}\right)\right]$

$= \frac{1}{\pi}\left[\cancel{\frac{-\pi^2\cos n\pi}{n}} + \cancel{\frac{2\cos n\pi}{n^3}} + \cancel{\frac{\pi^2\cos n\pi}{n}} - \cancel{\frac{2\cos n\pi}{n^3}}\right]$

$= 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$

$= \frac{1}{\pi}\left[x^2 \times \frac{\sin nx}{n} - 2x \times \frac{-\cos nx}{n^2} + 2x \frac{-\sin nx}{n^3}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\cancel{\frac{x^2\sin nx}{n}}^{0} + \frac{2x\cos nx}{n^2} - \cancel{\frac{2\sin nx}{n^3}}^{0}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left(\frac{2\pi\cos n\pi}{n^2} - \frac{2(-\pi)\cos n(-\pi)}{n^2}\right)$

$= \frac{1}{\pi}\left(\frac{2\pi\cos n\pi}{n^2} - \frac{-2\pi\cos n\pi}{n^2}\right)$

$= \frac{1}{\pi}\left(\frac{2\pi\cos n\pi}{n^2} + \frac{2\pi\cos n\pi}{n^2}\right)$

$= \frac{1}{\pi}\left(\frac{4\pi\cos n\pi}{n^2}\right)$

$= \frac{4\cos n\pi}{n^2}$

$= \frac{4(-1)^n}{n^2}$

Numerical Method (Unit-6)

(1) Algebraic function eg:- $x, x^2, x^3+x^4, \ldots$

x(2) Transcedental fun.

(a) Bisection Method -

(b) Newton-Raphson - N.R.

(a) Bisection Method - let $f(x)$ be a function.

$$f(0) = -ve = a$$
$$f(1) = -ve = b$$
$$f(2) = +ve = c$$

Bisection Root Lines between -ve to +ve.

Root Line betn [1,2]

$$x_1 = \frac{a+b}{2} = 3/2 = 1.5$$

$$f(x_1) = -ve$$

$$x_2 = \frac{1.5+2}{2} = 1.75$$

$$f(x_2) = +ve$$

Ex-1

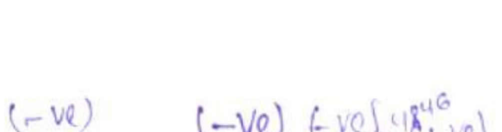
Find the root of the equation $x^4 - x - 10 = 0$ using the bisection method, correct up to 2 decimal places.

let, $f(x) = x^4 - x - 10$

$$f(0) = -10 \ (-ve)$$

$$f(1) = \cancel{1} - \cancel{1} - 10$$
$$= -10 \ (-ve)$$

$$f(2) = 2^4 - 2 - 10$$
$$= 16 - 2 - 10$$
$$= 16 - 12$$
$$= 4 \ (+ve)$$



$\therefore$ Root lies between [1 and 2] here, $a = 1, b = 2$

$$x_1 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5 \ (-ve)$$

$$f(1.5) = (1.5)^4 - 1.5 - 10$$
$$= -6.43$$

$$x_2 = \frac{1.5+2}{2} = 1.75 \quad (-ve)$$

$$f(1.75) = (1.75)^4 - (1.75) - 10$$
$$= -2.37$$

$$x_3 = \frac{1.75+2}{2} = 1.875 \quad (+ve)$$

$$f(1.875) = (1.875)^4 - (1.875) - 10$$
$$= 0.4846.$$

$$x_4 = \frac{\cancel{0.4846}\ 1.875+2}{2} = 1.8125 \quad (-ve)$$

$$f(1.8125) = (1.8125)^4 - (1.8125) - 10$$
$$= -1.0202$$

$$x_5 = \cancel{\frac{1.8125+2}{2}} \quad \frac{1.8125+1.875}{2}$$
$$= 1.84375$$

Hence, the root of $f(x)$ is 1.8.

(2) Newton - Raphson Method (NR Method)

Let, $f(x)$ be a function,
then, solve by newton - Raphson by using formula :-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ where } n = 0, 1, 2 \ldots\ldots \infty.$$

Ex-1
Find the root of the eqn $x^3 - 3x + 1 = 0$ by Newton-Raphson Method upto threee decimal places.

Let, $f(x) = x^3 - 3x + 1$.

$f(0) = 1$ (positive)

$f(1) = 1 - 3 + 1$
$= 2 - 3$
$= -1$ (Negative)

$f(2) = 8 - 6 + 1 = 3$ (positive)

$\therefore$ Root lies between $[1, 2]$

$$\boxed{x_0 = \frac{1+2}{2} = 1.5}$$

$$f'(x) = 3x^2 - 3.$$

$\therefore f(x_n) = x_n^3 - 3x_n + 1$

$f'(x_n) = 3x_n^2 - 3$.

By newton Rahpson Method :-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = \frac{x_n}{1} - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}$$

$$= \frac{3x_n^3 - 3x_n - x_n^3 + 3x_n - 1}{3x_n^2 - 3}$$

$$\boxed{x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 - 3}} \quad \text{——— (1)}$$

Put $n=0$ in equation '1'.

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 3}$$

$$= \frac{2(1.5)^3 - 1}{3(1.5)^2 - 3}$$

$$= 1.53$$

Put $n=1$ in equation '1':

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 3}$$

$$= \frac{2\times(1.53)^3 - 1}{3\times(1.53)^2 - 3}$$

$$= 1.53$$

Put $n=2$ in equation '1':-

$$x_3 = \frac{2x_2^3 - 1}{3x_2^2 - 3}$$

$$= \frac{2\times(1.53)^3 - 1}{3\times(1.53)^2 - 3}$$

$$= 1.53.$$

$\therefore$ So root of $x^3 - 3x + 1 = 0$ is 1.53.

(2) $f(x) = 3x^3 - 9x^2 + 8 = 0$. correct up to 3 decimal places.

~~$f(x_n)$~~ $f(0) = 8$ (positive)

$f(1) = 3 - 9 + 8$
$= 2$ (+ve)

$f(2) = 3\times 8 - 9\times 4 + 8$
$= 24 - 36 + 8$
$= 32 - 36$
$= -4$ (negative)

$f(3) = 81 - 81 + 8$
$= 8$ (positive)

The roots lies between $[2,3]$.

$$x_0 = \frac{2+3}{2} = 5/2 = 2.5$$

$f'(x) = 3\times 3x^2 - 9\times 2x$
$= 9x^2 - 18x$

$f(x_n) = 3x_n^3 - 9x_n^2 + 8$

$f'(x_n) = 9x_n^2 - 18x_n$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(3x_n^3 - 9x_n^2 + 8)}{9x_n^2 - 18x_n}$$

$$x_{n+1} = \frac{9x_n^3 - 18x_n^2 - 3x_n^3 + 9x_n^2 - 8}{9x_n^2 - 18x_n}$$

$$\boxed{x_{n+1} = \frac{6x_n^3 - 9x_n^2 - 8}{9x_n^2 - 18x_n}}$$

$n = 0.$

$$x_1 = \frac{6x_0^3 - 9x_0^2 - 8}{9x_0^2 - 18x_0}$$

$$= \frac{6\times(2.5)^3 - 9(2.5)^2 - 8}{9(2.5)^2 - 18(2.5)}$$

$$= 2.622.$$

$n = 1$

$$x_2 = \frac{6x_1^3 - 9x_1^2 - 8}{9x_1^2 - 18x_1}$$

$$= \frac{6\times(2.62)^3 - 9\times(2.62)^2 - 8}{9(2.62)^2 - 18(2.62)}$$

$$= 2.608.$$

$n = 2$

$$x_3 = \frac{6x_2^3 - 9x_2^2 - 8}{9x_2^2 - 18x_2}$$

$$= \frac{6(2.608)^3 - 9\times(2.608)^2 - 8}{9(2.608)^2 - 18(2.608)}$$

$$= 2.6079.$$

Numerical Method

(1) Find the Second approximation to the root corrected upto two decimal places of $x^3-5x+1=0$ by bisection method in $[2,3]$.

$f(x) = x^3-5x+1$

$x_1 = \frac{2+3}{2} = 5/2 = 2.5$ (positive)

$f(x_1) = (2.5)^3 - 5(2.5) + 1$

$= 4.125$

$x_2 = \frac{2.5 + 2}{2} = \cancel{3.5625}\ 2.25$ (Positive)

$f(x_2) = \cancel{(3.5625)^3 - 5(3.5625) + 1}$

$\cancel{= 28.400}$

$(2.25)^3 - 5(2.25) + 1$

$= 1.140$

$x_3 = \frac{2.25+2}{2} = 2.125$ (Negative)

$f(x_3) = (2.125)^3 - 5(2.125) + 1$

$= -0.02929$

$x_4 = \frac{2.125 + 2.25}{2} = 2.1875.$

(2) $f(x) = 5x^3 - 9x^2 + 8$

$f(0) = 8$

$f(1) = 5 - 9 + 8$

$= -4 + 8$

$= 4$

$f(2) = 5\times8 - 9\times4 + 8$

$= 40 - 36 + 8$

$= 4 + 8$

$= 12$

$f' = 5\times3x^2 - 9\times2x.$

$= 15x^2 - 18x$

Let, $\boxed{x_0 = 1}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{5x_n^3 - 9x_n^2 + 8}{15x_n^2 - 18x_n}$$

$$= \frac{15x_n^3 - 18x_n - 5x_n^3 + 9x_n^2 - 8}{15x_n^2 - 18x_n}$$

$$= \frac{10x_n^3 - 9x_n^2 - 8}{15x_n^2 - 18x_n}$$

$n=0$.

$$x_1 = \frac{10\times x_0^3 - 9\times x_0^2 - 8}{15x_0^2 - 18x_0}$$

$$= \frac{10\times 1^3 - 9\times 1^2 - 8}{15\times 1^2 - 18\times 1}$$

$$= \frac{10-9-8}{15-18} = \frac{10-17}{-3} = \frac{+7}{+3} = 7/3 = 2.333$$

$n=1$

$$x_2 = \frac{10\times x_1^3 - 9\times x_1^2 - 8}{15x_1^2 - 18\times x_1}$$

$$= \frac{10\times(2.33)^3 - 9\times(2.33)^2 - 8}{15(2.33)^2 - 18\times(2.33)}$$

$$= 1.7631$$

$n=2$

$$x_3 = \frac{10\times x_2^3 - 9\times x_2^2 - 8}{15\times x_2^2 - 18\times x_2}$$

$$= \frac{10\times(1.763)^3 - 9\times(1.763)^2 - 8}{15\times(1.763)^2 - 18\times(1.763)}$$

$$= 2.0774.$$

(3) $f(x) = x^3 - 3x + 1$

$f(0) = 1$

$f(1) = 1 - 3 + 1$
$= 2 - 3$
$= -1$

$f(2) = 8 - 6 + 1$
$= 2 + 1$
$= 3.$

The roots lies between 1 and 2.

$$x_0 = \frac{1+2}{2} = 3/2 = 1.5$$

$f(x_n) = x_n^3 - 3x_n + 1.$

$f'(x) = 3x^2 - 3.$

$f'(x_n) = 3x_n^2 - 3$

$$\Rightarrow x_{n+1} = \frac{x_n}{1} - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = \frac{}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = \frac{x_n}{1} - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}$$

$$\Rightarrow x_{n+1} = \frac{3x_n^3 - 3x_n - x_n^3 - 3x_n + 1}{3x_n^2 - 3}$$

$$\Rightarrow x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 - 3}$$

$n=0$

$$\Rightarrow x_1 = \frac{2x_0^3 + 1}{3x_0^2 - 3}$$

$$= \frac{2\times(1.5)^3 + 1}{3(1.5)^2 - 3}$$

$$= 2.0666$$

$\therefore n=1$

$$\Rightarrow x_2 = \frac{2x_1^3 + 1}{3x_1^2 - 3}$$

$$= \frac{2(2.0666)^3 + 1}{3(2.066)^2 - 3}$$

$$= 1.9023$$

$n=2$

$$\Rightarrow x_3 = \frac{2x_2^3 + 1}{3x_2^2 - 3}$$

$$= \frac{2(1.9023)^3 + 1}{3(1.9023)^2 - 3}$$

$$= 1.8797$$

(5) $f(x) = 2x^3 - 2x - 5 = 0$. (Newton-Raphson Method)

$f(0) = -5$

$f(1) = 2 - 2 - 5 = -5$

$f(2) = 2\times 8 - 2\times 2 - 5$
$= 16 - 4 - 5$
$= 16 - 9$
$= 7$

$x_0 = \frac{1+2}{2} = 3/2 = 1.5$

$f(x_n) = 2x_n^3 - 2x_n - 5$

$f'(x_0) = 2\times 3x^2 - 2 = 6x^2 - 2$.

$f'(x_n) = 6(x_n^2) - 2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

~~$n=0$. $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$~~

~~$= x_0$~~

$\Rightarrow x_{n+1} = \frac{x_n}{1} - \frac{(2x_n^3 - 2x_n - 5)}{6x_n^2 - 2}$

$\Rightarrow x_{n+1} = \frac{6x_n^3 - \cancel{2x_n} - 2x_n^3 + \cancel{2x_n} + 5}{6x_n^2 - 2}$

$\Rightarrow x_{n+1} = \frac{4x_n^3 + 5}{6x_n^2 - 2}$.

$n = 0$.

$x_1 = \frac{4\times x_0^3 + 5}{6\times x_0^2 - 2}$

$= \frac{4\times(1.5)^3 + 5}{6\times(1.5)^2 - 2}$

$= 1.6086$

$n = 1$

$x_2 = \frac{4\times x_1^3 + 5}{6\times x_1^2 - 2}$

$= \frac{4\times(1.60)^3 + 5}{6\times(1.60)^2 - 2}$

$= 1.6005$

$n = 2$

$x_3 = \frac{4\times x_2^3 + 5}{6\times x_2^2 - 2}$

$= \frac{4\times(1.6005)^3 + 5}{6\times(1.6005)^2 - 2}$

$= 1.60059$.

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n -$

(6) $f(x) = x^3 - 2x - 5$

$f(0) = -5$

$f(1) = 1 - 2 - 5$
$= -1 - 5$
$= -6$

$f(2) = 8 - 2\times2 - 5$
$= 8 - 4 - 5$
$= 4 - 5$
$= -1.$

$f(3) = 27 - 2\times3 - 5$
$= 27 - 6 - 5$
$= 27 - 11$
$= 16$

$x_0 = \frac{2+3}{2} = 2.5$ (positive).

$f(x_0) = (2.5)^3 - 2(2.5) - 5$
$= 5.625$

$x_1 = \frac{2.5+2}{2} = 2.25$ (positive)

$f(x_1) = (2.25)^3 - 2(2.25) - 5$
$= 1.690$

$x_2 = \frac{2.25+2}{2} = 2.125$ (positive)

$f(x_2) = (2.125)^3 - 2(2.125) - 5$
$= 0.3457$.

$x_3 = \frac{2.125+2}{2} = 2.0625$ (positive)

$f(x_3) = (2.0625)^3 - 2(2.0625) - 5$
$= 1.7111$

(7) $f(x) = \sqrt{10}$.
$x = \sqrt{10}$
$x^2 = 10$
$x^2 - 10 = 0$.

$f(x) = x^2 - 10$.
$f(0) = -10$.
$f(1) = 1 - 10$
$= -9$
$f(2) = 4 - 10$
$= -6$
$f(3) = 9 - 10$
$= -1$
$f(4) = 16 - 10$
$= 6$

$x_0 = \frac{3+4}{2}$
$= 7/2 = 3.5$

The roots lies between 3 and 4.

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_{n+1} = \frac{x_n}{1} - \frac{x_n^2 - 10}{2x_n}$

$x_{n+1} = \frac{2x_n^2 - (x_n^2 - 10)}{2x_n}$

$x_{n+1} = \frac{2x_n^2 - x_n^2 + 10}{2x_n}$

$x_{n+1} = \frac{x_n^2 + 10}{2x_n}$

$n = '0'$

$$x_1 = \frac{x_0^2 + 10}{2x_0} = \frac{3.5^2 + 10}{2 \times 3.5} = 1.92857$$

$n = '1'$

$$x_2 = \frac{x_1^2 + 10}{2x_1} = \frac{(1.928)^2 + 10}{2 \times (1.928)} = 3.557$$

$n = '2'$

$$x_3 = \frac{x_2^2 + 10}{2x_2} = \frac{(3.557)^2 + 10}{2 \times 3.557} = 3.181$$

$n = '3'$

$$x_4 = \frac{x_3^2 + 10}{2x_3} = \frac{(3.181)^2 + 10}{2(3.181)} = 3.162$$

$n = '4'$

$$x_5 = \frac{x_4^2 + 10}{2x_4} = \frac{(3.162)^2 + 10}{2 \times 3.162} = 3.162$$

(8) $x^3 - x - 4 = 0$

The roots lying between 1 and 2.

$f(x) = x^3 - x - 4$

$f(0) = -4$

$f(1) = 1^3 - 1 - 4 = -4$

$f(2) = 2^3 - 2 - 4 = 8 - 2 - 4 = 6 - 4 = 2$

$x_0 = \frac{1+2}{2} = 3/2 = 1.5$ (Negative)

$f(x_0) = (1.5)^3 - 1.5 - 4 = -2.125$

$x_1 = \frac{1.5 + 2}{2} = 1.75$ (Negative)

$f(x_1) = (1.75)^3 - 1.75 - 4 = -0.390$

$x_2 = \frac{1.75 + 2}{2} = 1.875$ (positive)

$f(x_2) = (1.875)^3 - 1.875 - 4 = 0.7167$

$$x_3 = \frac{1.875 + 1.75}{2} = 1.812 \text{ (positive)}$$

$f(x_3) = (1.812)^3 - 1.812 - 4 = 0.1374$

(9) $f(x) = \sqrt{2}$

$x = \sqrt{2}$

$x^2 = 2$

$x^2 - 2 = 0$

$f(x) = x^2 - 2$

$f(0) = -2$

$f(1) = 1 - 2 = -1$

$f(2) = 4 - 2 = 2$

the root lies between 1 and 2.

$x_0 = \frac{1+2}{2} = 3/2 = 1.5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = \frac{x_n}{1} - \frac{x_n^2 - 2}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{2x_n^2 - (x_n^2 - 2)}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{2x_n^2 - x_n^2 + 2}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{x_n^2 + 2}{2x_n}$$

$n = 0$.

$$x_1 = \frac{(1.5)^2 + 2}{2(1.5)}$$

$$= 1.41666$$

$n = 1$.

$$x_2 = \frac{(x_1)^2 + 2}{2x_1}$$

$$= \frac{(1.416)^2 + 2}{2 \times (1.416)}$$

$$= 1.414$$

$n = 2$

$$x_3 = \frac{(x_2)^2 + 2}{2x_2}$$

$$= \frac{(1.414)^2 + 2}{2(1.414)}$$

$$= 1.414$$

Finite Difference and Interpolation

Let, $Y = f(x) = x_0 + x_1 + x_2 + x_3 + \cdots + x_n$ (n+1 node)

$\Rightarrow Y = f(x)$ has (n+1) equispaced nodes.

$\Rightarrow h = \frac{x_n - x_0}{n}$

Types of difference -

(1) Shift operator (E)
(2) forward operator (Δ) (Delta)
(3) Backword operator (∇) (Nabla)

Shift operatore

Let, $Y = f(x) = x_0, x_1, x_2 \ldots x_n$ (n+1) equispaced.

$\Rightarrow$ then the shift operatore generally donated by "E".

$$\boxed{\Rightarrow E(f(x_i)) = f(x_i + h)}, \ i = 0, 1, 2 \ldots n.$$

$$E f(x_0) = f(x_0 + h)$$
$$E f(x_1) = f(x_1 + h)$$
$$E f(x_2) = f(x_2 + h)$$
$$\vdots$$
$$E f(x_n) = f(x_n + h)$$

Forcword operatore (Δ)

Let $f(x)$ be a function of (n+1), nodes with equal space difference.

$\Rightarrow$ Then the forword operator denoted by symbol Δ.

$$\Rightarrow \boxed{\Delta(f(x)) = f(x+h) - f(x)}$$

e.g :- $\Delta(f(x_0)) = f(x_0 + h) - f(x_0)$

$\Delta(f(x_1)) = f(x_1 + h) - f(x_1)$

$\vdots$

e.g :-

e.g:- Evaluate $\Delta^2(3e^x)$.

Sol^n $\Delta^2(3e^x)$

$= 3\Delta^2(e^x)$

$= 3\Delta(\Delta e^x)$

$= 3\Delta(e^{x+h}-e^x)$

$= 3\{\Delta e^{x+h}-\Delta e^x\}$

$= 3\{e^{x+2h}-e^{x+h}\}-\{e^{x+h}-e^x\}$

$= 3\{e^{x+2h}-e^{x+h}-e^{x+h}+e^x\}$

or

$\Delta^2(3e^x)$

$= 3\Delta^2(e^x)$

$= 3\Delta(\Delta e^x)$

$= 3\Delta(e^{x+h}-e^x)$

$= 3\Delta[e^x e^h - e^x]$

$= 3\Delta[e^x(e^h-1)]$

$= 3(e^h-1)(\Delta e^x)$

$= 3(e^h-1)(e^{x+h}-e^x)$

$= 3(e^h-1)[e^x e^h - e^x]$

$= 3(e^h-1)e^x(e^h-1)$

$= 3e^x(e^h-1)^2$

Ex-2 $\Delta(\tan^{-1}x)$

Sol.

$= \Delta(\tan^{-1}x)$

$= \tan^{-1}(x+h)-\tan^{-1}x$

$= \tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right)$

$= \tan^{-1}\left(\frac{h}{1+x^2+hx}\right)$

Answer

Q-3 Evaluate : $\Delta^n(a^x)$

Sol^n $\Delta^n(a^x)$

$= \Delta^{n-1}(\Delta a^x)$

$= \Delta^{n-1}(a^{x+h}-a^x)$

$= \Delta^{n-1}(a^x.a^h-a^x)$

$= \Delta^{n-1}\{a^x(a^h-1)\}$

$= (a^h-1)\Delta^{n-1}(a^x)$

$= (a^h-1)\Delta^{n-2}(\Delta a^x)$

$= (a^h-1)\Delta^{n-2}(a^{x+h}-a^x)$

$= (a^h-1)^2\Delta^{n-2}(a^x)$

$\Delta^n(a^x) = (a^h-1)^n.a^x$

Table of forward difference operator.

$$\boxed{\Delta\left[f(x_i)\right] = f(x_i+h) - f(x_i)} \quad i = 0, 1, 2 \cdots\cdots n$$

let, $y = f(x)$

x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	$f(x_0)$	$f(x_1)-f(x_0)$	$f(x_2)-f(x_1)-\{f(x_1)-f(x_0)\}$	
x_1	$f(x_1)$	$f(x_2)-f(x_1)$		
x_2	$f(x_2)$			

Ex-1 Form the forward difference table for the following data.

x	0	1	2	3	4
y	8	11	9	15	6

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	8	3	-5	13	-36
1	11	-2	8	-23	
2	9	6	-15		
3	15	-9			
4	6				

Write down the forward table if:

x	10	20	30	40
y	1.1	2.0	4.5	7.9

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
10	1.1	0.9	1.6	-0.7
20	2.0	2.5	0.9	
30	4.5	3.4		
40	7.9			

IMP

Form a table of forward difference for the function,
$f(x) = x^3 + 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$ calculate the table, to obtained $f(6)$.

$f(x) = x^3 + 5x - 7$

$Y_0 = (-1)^3 + 5(-1) - 7$ ~~$-1+25-7$~~

~~$= -30$~~

$= -1 - 5 - 7$

$= -13$

$Y_1 = -7$

$Y_2 = -1$

$Y_3 = 11$

$Y_4 = (3)^3 + 5 \times 3 - 7$

~~$= 9+15-7$~~ $27 + 15 - 7$

~~$= 9+8$~~ $= 35$

$= 77$

$Y_5 = 77$

$Y_6 = 143$

x	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$	$\Delta^6 Y$
-1	-13						
		6					
0	-7		0				
		6		6			
1	-1		6		0		
		12		6		0	
2	11		12		0		0
		24		6		0	
3	35		18		0		$Y_6 - 239$
		42		6		$Y_6 - 239$	
4	77		24		$Y_6 - 239$		
		66		$Y_6 - 233$			
5	143		$Y_6 - 209$				
		$Y_6 - 143$					
6	Y_6						

$\therefore Y_6 - 239 = 0$

$Y_6 = 239$ $[f(6)] = 239$.

IMP

Estimate the missing term in the following table.

x	0	1	2	3	4
y	1	3	9	—	81

Let the missing term Y_3.

x	y	Δy	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$
0	1				
		2			
1	3		4		
		6		$Y_3 - 19$	
2	9		$Y_3 - 15$		$124 - 4Y_3$
		$Y_3 - 9$		$(90 - 2Y_3) - (Y_3 - 15)$	
3	Y_3		$(81 - Y_3) - (Y_3 - 9)$		
		$81 - Y_3$			
4	81				

$81 - Y_3 - Y_3 + 9$
$= -2Y_3 + 90$
$= 90 - 2Y_3$

$90 - 2Y_3 - Y_3 + 15$
$= -3Y_3 + 105$
$= 105 - 3Y_3$

$105 - 3Y_3 - (Y_3 - 19)$
$= 105 - 3Y_3 - Y_3 + 19$
$= -4Y_3 + 124$
$= 124 - 4Y_3$

Here 4th order difference becomes zero.

$124 - 4Y_3 = 0$
$+4Y_3 = +124$
$Y_3 = \frac{124}{4}$
$= 31$

$\therefore$ Hence the missing term is 31.

(3) Backward difference operator

Let $Y = f(x) = x_0, x_1, x_2, \ldots \ldots x_n \rightarrow x_{(n+1)}$ node

$\Rightarrow$ Backward difference operator denoted by "∇".

$\Rightarrow \nabla f(x_i) = f(x_i) - f(x_i - h)$, $i = 0, 1, 2 \ldots \ldots n$.

Put $i = 0$

$\nabla f(x_0) = f(x_0) - f(x_0 - h)$

Put $i = 1$ $\nabla f(x_1) = f(x_1) - f(x_1 - h)$

$\vdots$

* Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$y_1 - y_0$		
x_1	y_1		$(y_2-y_1)-(y_1-y_0)$	
x_2	y_2	$y_2 - y_1$		
$\vdots$	$\vdots$			
x_n	y_n			

Relation between E, Δ & ∇.

~~Backward difference operator :- (∇)~~

(1) $\Delta = E - 1$

P.f By shift operator→

$$Ef(x) = f(x+h)$$

Again By forward operator-

$$\Delta f(x) = f(x+h) - f(x)$$
$$\Delta f(x) = E f(x) - f(x)$$
$$\Delta f(x) = f(x)[E-1]$$
$$\Delta f(x) = f(x)[E-1]$$
$$\Rightarrow \Delta = E - 1$$

(Proved)

(2) $\nabla = 1 - E^{-1}$

By Shift operator

$$Ef(x) = f(x+h)$$
$$\Rightarrow E^{-1}f(x) = f(x-h)$$

By definition of Backward difference operator:-

$$\nabla f(x) = f(x) - f(x-h)$$
$$\Rightarrow \nabla f(x) = f(x) - E^{-1} f(x)$$
$$\Rightarrow \nabla f(x) = f(x)[1 - E^{-1}]$$
$$\Rightarrow \nabla = [1 - E^{-1}]$$

(Proved)

$$\boxed{E^{-n} f(x_i) = f(x_i - nh)}$$

Find the value of $\Delta^n(e^x)$

$= \Delta^1 e^x$

$= e^{x+h} - e^x$

$= e^x . e^h - e^x$

$= e^x(e^h - 1)$

$\Delta^2 e^x$

$= \Delta . \Delta e^x$

$= \Delta [e^x(e^h-1)]$

$= (e^h - 1) [\Delta e^x]$

$= (e^h-1)(e^{x+h} - e^x)$

$= (e^h-1)(e^x . e^h - e^x)$

$= (e^h-1)[e^x(e^h-1)]$

$= e^x (e^h-1)^2$

⋮

$\Delta^n(e^x) = e^x(e^h-1)^n$

(2) $\Delta (\tan^{-1} x)$

$= \tan^{-1}(x+h) - \tan^{-1} x$

$= \tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right)$

$= \tan^{-1}\left(\frac{h}{1+x^2+hx}\right)$

① Find $\Delta^2(ab^x)$ where $h = 1$

$= \Delta^2(ab^x)$

$= \Delta . \Delta (ab^x)$

$= \Delta [ab^{x+h} - ab^x]$

$= \Delta [ab^{x+1} - ab^x]$

$= \Delta [ab^x . ab^1 - ab^x]$

$= \Delta [ab^x(ab^1 - 1)]$

$= (ab^1 - 1) \Delta (ab^x)$

$= (ab^1 - 1) [ab^{x+h} - ab^x]$

$= (ab^1 - 1) [ab^{x+1} - ab^x]$

$= (ab^1 - 1) [ab^x . ab^1 - ab^x]$

$= (ab^1 - 1) [ab^x(ab^1 - 1)]$

$= (ab^1 - 1)^2 ab^x$

$= ab^x (ab^1 - 1)^2$

③

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	2				
		2			
1	4		4		
		6		y_4-20	
2	10		y_4-16		$-4y_4+64$
		y_4-10		$-3y_4+44$	
3	y_4		$-2y_4+28$		
		$18-y_4$			
4	18				

$(18-y_4)-(y_4-10)$
$= 18-y_4-y_4+10$
$= -2y_4+28$

y_4-16-4
$= y_4-20$

$(-2y_4+28)-(y_4-16)$
$= -2y_4+28-y_4+16$
$= -3y_4+44$

$(-3y_4+44)-(y_4-20)$
$= -3y_4+44-y_4+20$
$= -4y_4+64$

$\Rightarrow -4y_4+64=0$
$\Rightarrow +4y_4=+64$
$\Rightarrow y_4=\frac{\cancel{64}\,16}{\cancel{4}}$
$\Rightarrow y_4=16$

$\therefore$ Missing teerm is 16.

④ $\Delta(ab^{cx})$
$= \Delta(ab^{cx})$
$= (ab^{cx+h})-(ab^{cx})$
$= (ab^{cx}\cdot ab^{h})-ab^{cx}$
$= ab^{cx}(ab^{h}-1)$

⑤ $\boxed{\Delta = E-1}$ $\boxed{\nabla = 1-E^{-1}}$

By shift operator:-
$Ef(x) = \cancel{\Delta}f(x+h)$
Again By forward operator -
$\Delta f(x) = f(x+h)-f(x)$
$\Delta f(x) = Ef(x)-f(x)$
$\Delta f(x) = f(x)[E-1]$
$\Rightarrow \boxed{\Delta = E-1.}$

$\boxed{\nabla = 1-E^{-1}}$
By shift operator:-
$Ef(x) = f(x+h)$
$\Rightarrow E^{-1}f(x) = f(x-h)$
Again By Backward operator:-
$\nabla f(x) = f(x)-f(x-h)$
$\nabla f(x) = f(x)-E^{-1}f(x)$
$\nabla f(x) = f(x)[1-E^{-1}]$
$\boxed{\nabla = 1-E^{-1}}$

⑥ $\Delta^n(e^x)$
$= \Delta^1(e^x)$
$= (e^{x+h})-(e^x)$
$= e^x\cdot e^h - e^x$
$= e^x(e^h-1)$
$\Delta^n e^x$
$= e^x(e^h-1)^n$

(7) $\Delta = E - 1$.

By Shift operator:-

$$E f(x) = f(x+h)$$

Again By forward operator:-

$$\Rightarrow \Delta f(x) = f(x+h) - f(x)$$

$$\Rightarrow \Delta f(x) = E f(x) - f(x)$$

$$\Rightarrow \Delta f(x) = f(x)\left[E - 1\right]$$

$$\Rightarrow \Delta = E - 1 \text{ (proved)}$$

Interpolation

Let, $y = f(x) = x_0, x_1, x_2 \cdots\cdots x_n \rightarrow (n+1)$ equispaced nodes.

(1) Lagrange Linear Interpolation.

$$p(x) = \cancel{(x_1 - x_0)(x_2 - x_0) \cdots\cdots (x_n - x_0)}$$

$$\frac{(x - x_1)(x - x_2) \cdots\cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots\cdots (x_0 - x_n)} \times f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2) \cdots\cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots\cdots (x_1 - x_n)} \times f(x_1)$$

$$+ \cdots\cdots + \frac{(x - x_0)(x - x_1)(x - x_2) \cdots\cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots\cdots (x_n - x_{n-1})} \times f(x_n)$$

Interepolation

Eg-1 Using lagrange interpolation to find the value of y when $x=10$. If the following values of x and y are given.

x : 5 6 9 11
y = 12 13 14 16

Soln

Here $x_0=5$, $x_1=6$, $x_2=9$, $x_3=11$

$f(x_0)=12$, $f(x_1)=13$, $f(x_2)=14$, $f(x_3)=16$

By lagrange Interpolation

$$p(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)$$

$$= \frac{(10-6)(10-9)(10-11)}{(9-5)(9-6)(9-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13 + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14$$

$$+ \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= \left(\frac{4 \times 1 \times -1}{4 \times 3 \times -2} \times 12\right) + \left(\frac{5 \times 1 \times -1}{1 \times 3 \times -5} \times 13\right) + \left(\frac{5 \times 4 \times -1}{4 \times 3 \times -2} \times 14\right) + \left(\frac{5 \times 4 \times 1}{6 \times 5 \times 2} \times 16\right)$$

$$= \left(\frac{+4}{+24} \times 12\right) + \left(\frac{+5}{+15} \times 13\right) + \left(\frac{+20}{+24} \times 14\right) + \left(\frac{20}{60} \times 16\right)$$

$$= 2 + 4.33 + 11.666 + 5.33$$

$$= 14.66$$

Newton divided difference

Let, $y = f(x) = x_0, x_1, x_2 \dots\dots\dots x_n \rightarrow (n+1)$ node.

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$\therefore p(x) = f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3]$$

Newton divided difference Table

x	$f(x)$	1st div. diff	2nd div. diff	3rd div. diff	4th div. diff
x_0	$f(x_0)$				
		$f[x_0, x_1]$			
			$f[x_0, x_1, x_2]$		
x_1	$f(x_1)$			$f[x_0, x_1, x_2, x_3]$	
		$f[x_1, x_2]$			$f[x_0, x_1, x_2, x_3, x_4]$
			$f[x_1, x_2, x_3]$		
x_2	$f(x_2)$			$f[x_1, x_2, x_3, x_4]$	
		$f[x_2, x_3]$			
			$f[x_2, x_3, x_4]$		
x_3	$f(x_3)$				
		$f[x_3, x_4]$			
x_4	$f(x_4)$				

Ex-1

Find a divided difference for a function $f(x) = x^2+2x+2$ whose arguments are, 1, 2, 4, 7 & 10.

Soln

$x_0 = 1$
$f(x_0) = x^2+2x+2$
$= (1)^2+2\cdot1+2$
$= 1+2+2$
$= 5$

$x_1 = 2$
$f(x_1) = x^2+2x+2$
$= 2^2+2\cdot2+2$
$= 4+4+2$
$= 10$

$x_2 = 4$
$f(x_2) = x^2+2x+2$
$= 4^2+2\cdot4+2$
$= 16+8+2$
$= 26$

$x_3 = 7$
$f(x_3) = x^2+2x+2$
$= 7^2+2\cdot7+2$
$= 49+14+2$
$= 65$

$x_4 = 10$
$f(x_4) = x^2+2x+2$
$= 10^2+2\cdot10+2$
$= 100+20+2$
$= 122$

x	$f(x)$	1st div. diff	2nd div. diff	3rd div. diff	4th div. diff
1	5				
		5			
2	10		1		
		8		0	
4	26		1		0.
		13		0	
7	65		1		
		19			
10	122				

(1) Use the Lagrange's interpolation formula to fit a polynomial to the data

x : 3 2 1 -1
y : 3 12 15 -21

$x_0 = 3$ $\quad x_1 = 2$ $\quad x_2 = 1$ $\quad x_3 = -1$
$f(x_0) = 3$ $\quad f(x_1) = 12$ $\quad f(x_2) = 15$ $\quad f(x_3) = -21$.

(2) Use Lagrange's interpolation formula to fit a polynomial to the given data.

x	0	1	3
$f(x)$	1	3	55

$x_0 = 0$, $\quad x_1 = 1$ $\quad x_2 = 3$
$f(x_0) = 1$ $\quad f(x_1) = 3$ $\quad f(x_2) = 55$

$$p(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)} \times 1 + \frac{(x-0)(x-3)}{(1-0)(1-3)} \times 3 + \frac{(x-0)(x-1)}{(3-0)(3-1)} \times 55$$

$$= \frac{(x-1)(x-3)}{(-1)(-3)} + \frac{x(x-3)}{1\times(-2)} \times 3 + \frac{x(x-1)}{3\times 2} \times 55$$

$$= \frac{x(x-3)-1(x-3)}{3} + \frac{x^2-3x}{-2} \times 3 + \frac{x^2-x}{6} \times 55$$

$$= \frac{x^2-3x-x+3}{3} + \frac{3x^2-9x}{-2} + \frac{55x^2-55x}{6}$$

$$= \frac{x^2-3x-x+3}{3} + \frac{-3x^2+9x}{2} + \frac{55x^2-55x}{6}$$

$$= \frac{2(x^2-3x-x+3)+3(-3x^2+9x)+55x^2-55x}{6}$$

$$= \frac{2(x^2-4x+3)-9x^2+27x+55x^2-55x}{6}$$

$$= \frac{2x^2-8x+6-9x^2+27x+55x^2-55x}{6}$$

$$= \frac{2x^2-9x^2+55x^2-8x+27x-55x+6}{6}$$

$$= \frac{48x^2-36x+6}{6}$$

$$= \frac{\cancel{6}(8x^2-6x+1)}{\cancel{6}}$$

$$= 8x^2-6x+1$$

(3) Find f(x) when $x = 15$

x	3	7	11	19
$f(x)$	42	43	47	60

$x_0 = 3 \quad x_1 = 7 \quad x_2 = 11 \quad x_3 = 19$

$f(x_0) = 42 \quad f(x_1) = 43 \quad f(x_2) = 47 \quad f(x_3) = 60$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)$$

$$= \frac{(15-7)(15-11)(15-19)}{(3-7)(3-11)(3-19)} \times 42 + \frac{(15-3)(15-11)(15-19)}{(7-3)(7-11)(7-19)} \times 43 +$$

$$\frac{(15-3)(15-7)(15-19)}{(11-3)(11-7)(11-19)} \times 47 + \frac{(15-3)(15-7)(15-11)}{(19-3)(19-7)(19-11)} \times 60$$

$$= 10.5 + (-43) + 70.5 + 15$$

$$= 53$$

Find the cubic polynomial which takes to following values:-

x:	0	1	2	3
$f(x)$:	1	2	1	10

Hence evaluate $f(4)$.

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

Hence,

$\therefore f(4) = 12$.

(5) Lagrange's interpolation formula :-

$$\Rightarrow p(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x-x_2)} f(x_3)$$

Newton's forward Interpolation :-

→ A polynomial $p(x)$ of an interpolation in the inteval $[a,b]$.

→ $p(x) = x_0, x_1, x_2 \cdots\cdots x_n$ $(n+1)$ equidistance node.

→ Newton's forward formula :-

$$P(x) = f(x_0) + \frac{(x-x_0)}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f_0 + \cdots\cdots + \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{n!h^n}\Delta^n f_0$$

Eg:-1

$y(75) = 246$, $y(80) = 202$, $y(85) = 118$, $y(90) = 40$ find $y(79)$?

Here, $x_0 = 75$, $x_1 = 80$, $x_2 = 85$, $x_3 = 90$

$y_0 = 246$, $y_1 = 202$, $y_2 = 118$, $y_3 = 40$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
75	246			
		-44		
80	202		-40	
		-84		46
85	118		6	
		-78		
90	40			

$h = 85 - 80$
$= 5.$

$$y(79) = f(x_0) + \frac{(x-x_0)}{1!h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f_0 + \frac{(x-x_0)(x-x_1)(x-x_2)}{3!h^3}\Delta^3 f_0$$

$$= 246 + \frac{(79-75)}{1\times 5}\times(-44) + \frac{(79-75)(79-80)}{2\times 25}\times(-40) + \frac{(79-75)(79-80)(79-85)}{6\times 125}\times(46)$$

$$= 246 + (-35.2) + 3.2 + 1.472$$

$\cancel{= 44.472}$ $= 215.472$

* Find a cubic polynomial which takes the following values.

x : 0 1 2 3

$f(x)$: 1 2 1 10

Solⁿ let's make a Newton's table :-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

Here, $h = 1$

By Newton's forward formula :-

$$p(x) = f(x_0) + \frac{(x-x_0)}{1!h}\Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f(x_0) + \frac{(x-x_0)(x-x_1)(x-x_2)}{3!h^3}\Delta^3 f(x_0)$$

$$= 1 + \frac{(x-0)}{1}\times 1 + \frac{(x-0)(x-1)}{2\times 1}\times -2 + \frac{(x-0)(x-1)(x-2)}{6\times 1}\times 12$$

$$= 1 + x + \frac{x(x-1)}{\cancel{2}}\times -\cancel{2} + \frac{x(x-1)(x-2)}{\cancel{6}}\times \cancel{12}\,2$$

$$= 1 + x - (x^2 - x) + (x^2 - x)(x-2)\times 2$$

$$= 1 + x - x^2 + x + (x^3 - 2x^2 - x^2 + 2x)\times 2$$

$$= 1 + 2x - x^2 + (x^3 - 3x^2 + 2x)\times 2$$

$$= 1 + 2x - x^2 + 2x^3 - 6x^2 + 4x$$

$$= 2x^3 - 7x^2 + 6x + 1$$

Newton's Backward interpolation -

Let, $y = f(x) = x_0, x_1, x_2 \ldots\ldots x_n$ $(n+1)$ equispace note

$$p(x) = f(x_n) + \frac{(x-x_n)}{1!h}\nabla f(x_n) + \frac{(x-x_n)(x-x_{n-1})}{2!h^2}\nabla^2 f(x_n) + \frac{(x-x_n)(x-x_{n-1})(x-x_{n-2})}{} \ldots\ldots \frac{(x-x_1)}{n!h^n}\nabla^n f(x_n)$$

Q) Apply Newton's Backword formula to find a polynomial of degree three which include the following x, y points:-

x:	3	4	5	6
$f(x)$:	6	24	60	120

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
3	6			
		18		
4	24		18	
		36		6
5	60		24	
		60		
6	120			

$x_n = 6,\ h = 1$.

$$p(x) = 120 + \frac{(x-6)}{1\times 1}\times 60 + \frac{(x-6)(x-5)}{2\times 1^2}\times 24 + \frac{(x-6)(x-5)(x-4)}{6\times 1^3}\times 6$$

$$= 120 + 60x - 360 + \frac{x^2-5x-6x+30}{\cancel{2}}\times \overset{12}{\cancel{24}} + \frac{(x^2-5x-6x+30)(x-4)}{\cancel{6}}\times\cancel{6}$$

$$= 120 + 60x - 360 + (x^2-11x+30)\,12 + (x^2-11x+30)(x-4)$$

$$= 120 + 60x - 360 + 12x^2 - 132x + 360 + x^3 - 11x^2 + 30x - 4x^2 + 44x - 120$$

$$= \cancel{120} + 60x - \cancel{360} + 12x^2 - 132x + \cancel{360} + \dots$$

$$= x^3 + 12x^2 - 11x^2 - 4x^2 + 60x - 132x + 30x + 44x$$

$$= x^3 - 3x^2 + 2x$$

Numerical Integration -

$$\int_a^b f(x)\,dx = \int_a^b y\,dx$$

Space diff. $h = \frac{b-a}{n}$.

$x_0 = a$ (lower limit) $\longrightarrow y_0 = f(x_0)$

$x_1 = x_0 + h \longrightarrow y_1 = f(x_1)$

$x_2 = x_0 + 2h \longrightarrow y_2 = f(x_2)$

$x_3 = x_0 + 3h \longrightarrow y_3 = f(x_3)$

$\vdots$

$x_n = x_0 + nh = b$ (upper limit) $\quad y_n = f(x_n)$

(1) Trapezoidal Rule
(2) Simpson's $\frac{1}{3}$rd Rule
(3) Simpson's $\frac{3}{8}$th Rule
(4) Weddle's Rule

} Newton's cotes formula (2 mark)

Newton's Integration -

Newton's cotes formula :-

$$\lambda_n \text{ or } R_n = \frac{(-1)^{n-k} \cdot n}{k!(n-k)!} \int_0^n s(s-1)(s-2) \cdots (s-n)\,ds$$

IMP.

(a) Trapezoidal Rule ($n=1$) (This rule is called trapezoidal rule because it is the sum of areas of trapezium under the curve)

$$I = \int_a^b f(x)\,dx = \int_a^b y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \cdots y_{n-1})\right]$$

[Here for every value of 'n']

(b) Simpson's $\frac{1}{3}$ Rule ($n=2$)

$$I = \int_a^b y\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots y_{n-2}) + 2(y_2 + y_4 + y_6 \cdots)\right]$$

Here 'n' is multiple of '2'.

Ex-1

$\int_0^6 \frac{1}{1+x^2}dx$ by using (i) Trapezoidal Rule (ii) Simpson's $\frac{1}{3}$ Rule.

Sol'n Given $\int_0^6 \frac{1}{1+x^2}dx$

$h = \frac{6-0}{6} = 1$

$y = f(x) = \frac{1}{1+x^2}$

To compute the values :-

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x :	0	1	2	3	4	5	6
y :	1	$1/2$	$1/5$	$1/10$	$1/17$	$1/26$	$1/37$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) Trapezoidal Rule

$I = \int_0^6 \frac{1}{1+x^2}dx$

$= \frac{1}{2}\left[(1+\frac{1}{37}) + 2(\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26})\right]$

$= 1.410798581$.

(ii) Simpson's $\frac{1}{3}$ Rule

$I = \int_0^6 \frac{1}{1+x^2}dx$

$= \frac{1}{3}\left[(1+\frac{1}{37}) + 4\times(\frac{1}{2}+\frac{1}{10}+\frac{1}{26}) + 2\times(\frac{1}{5}+\frac{1}{17})\right]$

$= 1.366173413$.

Q) Calculate by Simpson's Rule an approximate value of $\int_{-3}^{3} x^4 dx$ by taking seven equidistance nodes. Compare it with the exact value & the value obtained by using Trapezoidal Rule.

Sol^n Given $\int_{-3}^{3} x^4 dx$

$h = \frac{3-(-3)}{6} = \frac{3+3}{6} = 6/6 = 1.$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x :	-3	-2	-1	0	1	2	3
y :	81	16	1	0	1	16	81
	y_0	y_1	y_2	y_3	y_4	y_5	y_6.

Simpson's Rule

$I = \int_{-3}^{3} x^4 dx$

$= 1/3 \left[(81+81) + 4\times(16+0+16) + 2(1+1)\right]$

$= 1/3 \left[162 + 4\times(32) + 2\times(2)\right]$

$= 1/3 \left[162 + 128 + 4\right]$

$= 98$

Trapezoidal Rule

$I = \int_{-3}^{3} x^4 dx$

$= 1/2 \left[(81+81) + 2\times(16+1+0+1+16)\right]$

$= 1/2 \left[162 + 2\times(34)\right]$

$= 1/2 \left[162 + 68\right]$

$= 115$

∴ Trapezoidal doesn't give exact value.

exact value

$\int_{-3}^{3} x^4 dx$

$= \left(\frac{x^5}{5}\right)_{-3}^{3}$

$= \frac{243}{5} - \frac{-243}{5}$

$= \frac{243}{5} + \frac{243}{5}$

$= \frac{243+243}{5}$

$= \frac{486}{5}$

$= 97.2$

Difference.

$115 - 97.2$

$= 17.8$

* $\int_0^2 x^2\,dx$ using trapezoidal Rule taking $h = 1$

$h = 1$

$\int_0^2 x^2\,dx$

$y = f(x) = x^2$

	x_0	x_1	x_2
x :	0	1	2
y :	0	1	4
	y_0	y_1	y_2

$= \int_0^2 x^2\,dx$

$= \frac{h}{2}\left[(0+4) + 2(1)\right]$

$= \frac{1}{2}\left[4 + 2\right]$

$= \frac{6}{2} = 3$

* Find $\int_0^5 \frac{1}{4x+5}\,dx$ using Simpson $\frac{1}{3}$ Rule using 10 sub intervals.

$= y = f(x) = \frac{1}{4x+5}$

$h = 0.5 = \frac{1}{2}$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x :	0	0.5	1	1.5	2	2.5	3	3.5	4	[illegible]	5
y :	$\frac{1}{5}$	0.1428	$\frac{1}{9}$	0.0909	$\frac{1}{13}$	0.066	$\frac{1}{17}$	0.052	$\frac{1}{21}$	[illegible]	$\frac{1}{25}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8		y_{10}

$$= \int_0^{5} \frac{1}{4x+5} dx$$

$$= \frac{0.5}{3}\left[\left(\tfrac{1}{5}+\tfrac{1}{25}\right) + 4\times(0.1428+0.0909+0.066+0.052+0.043) + 2\times\left(\tfrac{1}{9}+\tfrac{1}{13}+\tfrac{1}{17}+\tfrac{1}{21}\right)\right]$$

$= 0.40129$

Q) Define Numerical Integration and state Trapezoidal Rule.

=> Numerical Integration – Numerical Integration is the process of computing the value of a definite integral from the tabulated value of the integrand.

$$I = \int_a^b f(x)dx = \int_a^b y\,dx = \frac{h}{2}\left[(y_0+y_n)+2\times(y_1+y_2+y_3+\cdots\cdots+y_{n-1})\right]$$

Q) Using Simpson's $\frac{1}{3}$rd rule & taking $h=1$ evaluate $\int_0^6 \frac{dx}{1+x}$.

$h = 1$

$y = f(x) = \frac{1}{1+x}$

		x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	:	0	1	2	3	4	5	6
y	:	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
		y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$= \int_0^6 \frac{dx}{1+x}$$

$$= \frac{1}{3}\left[\left(1+\frac{1}{7}\right)+4\times\left(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}\right)+2\times\left(\tfrac{1}{3}+\tfrac{1}{5}\right)\right]$$

$= 1.958730159$

Q) State Trapezoidal Rule. why this is called Trapezoidal Rule.

$$I = \int_a^b f(x)\,dx = \int_a^b y\,dx = \frac{h}{2}\left[(y_0+y_n) + 2\times(y_1+y_2+y_3+\cdots\cdots y_{n-1})\right]$$

This rule is called trapezoidal rule because it is the sum of area of trapezium under the curve.

Q) Evaluate $\int_{2.5}^{4} \ln x\,dx$ using Trapezoidal rule with 5 subintervals.

Q) Evaluate $\int_0^6 \frac{dx}{4x+5}$ using Simpson's $\frac{1}{3}$ Rule correct up to 3 place of decimal taking $h=1$.

$h=1$

$y = f(x) = \frac{1}{4x+5}$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x:	0	1	2	3	4	5	6
y:	$\frac{1}{5}$	$\frac{1}{9}$	$\frac{1}{13}$	$\frac{1}{17}$	$\frac{1}{21}$	$\frac{1}{25}$	$\frac{1}{29}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$= \int_0^6 \frac{dx}{4x+5}$$

$$= \frac{1}{3}\left[\left(\frac{1}{5}+\frac{1}{29}\right)+4\times\left(\frac{1}{9}+\frac{1}{17}+\frac{1}{25}\right)+2\times\left(\frac{1}{13}+\frac{1}{21}\right)\right]$$

$$= 0.4411018566.$$

Q) Find $\int_0^5 \frac{1}{4x+5}dx$ using Simpson's 1/3rd Rule using

Q) Evaluate $\int_0^6 \frac{1}{1+x^2}dx$ by using Simpson's 1/3 Rule and compare the result with its actual value.

$y = f(x) = \frac{1}{1+x^2}$

$h = \frac{6-0}{6} = 6/6 = 1$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x :	0	1	2	3	4	5	6
y :	1	1/2	1/5	1/10	1/17	1/26	1/37
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$= \int_0^6 \frac{1}{1+x^2}dx$$

$$= \frac{1}{3}\left[\left(1+\frac{1}{37}\right)+4\times\left(\frac{1}{2}+\frac{1}{10}+\frac{1}{26}\right)+2\times\left(\frac{1}{5}+\frac{1}{17}\right)\right]$$

$$= 1.366173413$$

Q) Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using Simpson's 1/3 rule taking $h = 1/4$.

$h = 1/4 = 0.25$

$y = f(x) = \frac{1}{1+x^2}$

	x_0	x_1	x_2	x_3	x_4
x :	0	0.25	0.5	0.75	1
y :	1	0.9411	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

$$\int_0^1 \frac{1}{1+x^2}$$

$$= \frac{0.25}{3}\left[(1+0.5)+4\times(0.9411+0.64)+2\times(0.8)\right]$$

$$= 0.7853666667$$

Q) Define Numerical integration and state two rules fore this purpose.

Numerical Integration is the process of computing the value of a definite integral from the tabulated value of the integrand.

Trapezoidal

$$I = \int_a^b f(x)\,dx = \int_a^b y\,dx = h/2\left[(y_0+y_n)+2\times(y_1+y_2+y_3+\cdots\cdots+y_{n-1})\right]$$

Simpson

$$I = \int_a^b f(x)\,dx = \int_a^b y\,dx = h/3\left[(y_0+y_n)+4\times(y_1+y_3+\cdots+y_{n-2})+2\times(y_2+y_3+\cdots y_{n-1})\right]$$

Q) Find $\int_0^5 \frac{1}{4x+5}\,dx$ using Simpson's $1/3$rd Rule using.

$$f(x) = y = \frac{1}{4x+5}.$$

	x_0	x_1	x_2	x_3	x_4	x_5
x:	0	1	2	3	4	5
y:	0.2	0.111	0.0769	0.0588	0.0476	0.0[illegible]
	y_0	y_1	y_2	y_3	y_4	y_5

$$h = \frac{5-0}{5} = 5/5 = 1$$

~~a) Find~~ $= \int \frac{1}{4x+5}$

$$= \frac{1}{3}\left[(0.2+0.04)+4\times(0.111+0.0588)+2\times(0.0769+0.0476)\right]$$

$$= 0.3894$$